Stability of critical shapes for the drag minimization problem in Stokes flow

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Abstract

We study the stability of some critical (or equilibrium) shapes in the minimization problem of the energy dissipated by a fluid (*i.e.* the drag minimization problem) governed by the Stokes equations. We first compute the shape derivative up to the second order, then provide a sufficient condition for the shape Hessian of the energy functional to be coercive at a critical shape. Under this condition, the existence of such a local strict minimum is then proved using a precise upper bound for the variations of the second order shape derivative of the functional with respect to the coercivity and differentiability norms. Finally, for smooth domains, a lower bound of the variations of the drag is obtained in terms of the measure of the symmetric difference of domains.

Keywords: stability of critical shape, drag minimization, optimal profiles, shape calculus, shape Hessian, Stokes equations.
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1 Introduction

In fluid mechanics, the study of the minimization of the drag of a body in a fluid (*i.e.* the computation of optimal profiles) is a very popular problem. A typical application is the study of the geometry of blunt bodies in flow at low Reynolds numbers (see [17]). In this work, the parameter is the shape of the body immersed in the fluid and we are interested in studying the stability of this shape optimization problem.

In an optimization problem, the study of the stability consists to know if a critical shape of a functional is a local strict minimizer. This point is particularly crucial in order to make numerical simulations: if the problem is unstable, regularization is required in the numerical minimization of the functional (see for example [2, 3]). The question of stability is addressed with the second order derivative of the functional at a critical point. The notion of derivative used in this work is the now classic shape calculus (Hadamard's approach) presented for example in [15, 18, 19, 21].

The strategy we follow is the following: a first step is to prove the existence of second order shape derivatives. It is classically obtained through an implicit function theorem: this is due to Simon [20] for Stokes equations and to Bello *et al.* [5, 6] for the Navier-Stokes equations). In a second step, we obtain the Euler-Lagrange equation, then we compute the shape Hessian. These shape derivatives of the drag (at least at the first order) was computed by some authors (see for example [20] for the Stokes equations and [6, 4] for the Navier-Stokes equations). However, in this work, we need a different expression of the

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shape Hessian (in [20]). Then, we use the Euler-Lagrange equations and some computations to derive a sufficient condition of positivity of the shape Hessian computed at a critical shape.

In a third step, we prove the stability of the minimization. In finite dimension, the knowledge of the sign of the Hessian permits to fully answer the question of stability using Taylor-Young expansion. However, in shape optimization or in infinite dimension, the two norms discrepancy problem can occur: the coercivity norm is often weaker than the differentiability norm. Let us refer to the paper [11] by Descloux for a concrete example of such a situation known as the magnetic shaping problem: on this example, the coercivity at a critical point holds in the $\mathrm{H}^{1/2}$ norm while the differentiability holds in $\mathcal{C}^2$ topology. Since the quantity $o(\|\cdot\|_{\mathcal{C}^2})$ is not smaller than $C\|\cdot\|_{\mathrm{H}^{1/2}}^2$, the classical argument using the Taylor-Young formula does not insure that this critical point is a local strict minimum. A method to overcome this problem is given by Dambrine *et al.* in [10, 9]. The key is a precise estimate of the variations of the second order shape derivative of the functional with respect to the coercivity and differentiability norms around a critical shape. In [10, 9], the authors study Poisson's equation or the case of a strictly and uniformly elliptic operator. Here, we adapt these methods to the drag minimization problem for a Stokes flow. Finally, in a fourth and last step, we derive a precise version of the minimality inequality: we provide a lower bound of the variations of the drag in term of a geometrical quantity following new ideas introduced by Fusco *et al.* [13, 1].

The paper is organized as follows. We first define the studied problem: we define the shape functional under consideration and the used domain perturbations. In particular, we recall some geometrical constructions from [10, 9]. In Section 3, following Hadamard's approach, we use shape calculus to prove the coercivity of the shape Hessian of the considered functional at an equilibrium shape in the $\mathrm{H}^{1/2}$ norm. Then, in Section 4, we state the main result of this work which is a stability result for the drag minimization problem. It claims that a critical shape can be a local strict minimum if a given criterion is satisfied. In Section 5, we translate our result in a more explicit estimation in purely geometrical quantities. Finally, we conclude the paper in Section 6 mentioning some possible extensions and prospects concerning the problem we study here.

2 The problem setting

2.1 General notations

Let us introduce the notations that we adopt in this paper. For an open set $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3), we denote by $\mathrm{L}^p(\Omega)$, $\mathrm{W}^{m,p}(\Omega)$ and $\mathrm{H}^s(\Omega)$ the usual Lebesgue and Sobolev spaces. We note in bold the vectorial functions and spaces: $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$, $\mathbf{H}^s(\Omega)$, etc. Moreover, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$, the space $\mathcal{C}^k(\Omega)$ is defined as the set of functions having continuous derivatives up to order k in Ω and $\mathcal{C}^{k,\alpha}(\Omega)$ denotes the usual Hölder space. We denote by $|\Omega|$ the measure of Ω. Moreover, $\mathbf{n}$ represents the external unit normal to $\partial\Omega$, and for a smooth enough function u, we note $\partial_{\mathbf{n}}u$ the normal derivative of u. Finally, we define the space

$$\mathrm{L}_0^2(\Omega) := \left\{ p \in \mathrm{L}^2(\Omega), \int_\Omega p = 0 \right\}.$$

2.2 Tubular coordinates and normal representation of a perturbed domain

Let us recall some basic facts from differential geometry and fix the notations. We follow in this paragraph the constructions and proof of [10]. Let Ω be a smooth bounded open set and let $\mathbf{n}$ denote the outer unit normal field to $\partial\Omega$. There is a non negative real $\rho(\Omega)$, such that the application $T_{\partial\Omega}$ defined by

$$
\begin{array}{rccc}
T_{\partial\Omega} & : \partial\Omega \times \mathbb{R} & \to & \mathbb{R}^d \\
& (M, h) & \mapsto & M + h\,\mathbf{n}(M)
\end{array}
$$

is a local diffeomorphism from $\partial\Omega \times (-\rho(\Omega), \rho(\Omega))$ on an open tubular neighborhood of $\partial\Omega$ we will denote $\mathcal{T}_{\partial\Omega}$. This property expresses the fact that any point x in $\mathcal{T}_{\partial\Omega}$ has a unique orthogonal projection $p_{\partial\Omega}(x)$ on $\partial\Omega$ and that the relation

$$
x = p_{\partial\Omega}(x) + h(x)\,\mathbf{n}(p_{\partial\Omega}(x)) \quad \text{where} \quad |h(x)| = \|x - p_{\partial\Omega}(x)\|,
$$

holds. Notice that h is uniquely defined.

Let us recall the definition of the Micheletti's distance on the set S of $\mathcal{C}^{2,\alpha}$ bounded domains of $\mathbb{R}^d$ that are diffeomorphic to Ω. The idea is to use the Banach imbedding of diffeomorphisms in $\mathcal{C}^{2,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$. For Ω_1 and Ω_2 in S, set

$$
\mathrm{d}_{2,\alpha}(\Omega_1, \Omega_2) := \inf \left(\|\boldsymbol{\Theta} - \mathbf{I}\|_{2,\alpha} + \|\boldsymbol{\Theta}^{-1} - \mathbf{I}\|_{2,\alpha} \right)
$$

where the infimum is taken over all the diffeomorphisms $\boldsymbol{\Theta} \in \mathcal{C}^{2,\alpha}(\mathbb{R}^d)$ mapping Ω_1 onto Ω_2. In particular, if $\boldsymbol{\Theta}$ is a perturbation of the identity such that

$$
\|\boldsymbol{\Theta} - \mathbf{I}\|_{2,\alpha} + \|\boldsymbol{\Theta}^{-1} - \mathbf{I}\|_{2,\alpha} < \rho(\Omega),
$$

any point of the boundary $\partial\Omega_{per}$ of the perturbed domain $\Omega_{per} := \boldsymbol{\Theta}(\Omega)$ lays in the tubular neighborhood $\mathcal{T}_{\partial\Omega}$ of $\partial\Omega$ so that it can be described in terms of normal deformations: for all $x \in \partial\Omega$

$$
\boldsymbol{\Theta}(x) = \boldsymbol{R}_{\boldsymbol{\Theta}}(x) + \mathrm{d}_{\partial\Omega}(\boldsymbol{\Theta}(x))\,\mathbf{n}(\boldsymbol{R}_{\boldsymbol{\Theta}}(x)),
$$

where $\boldsymbol{R}_{\boldsymbol{\Theta}}$ is a diffeomorphism from $\partial\Omega$ into itself and $\mathrm{d}_{\partial\Omega}$ is the signed distance to $\partial\Omega$. Note that $\widetilde{\mathbf{n}} := \nabla \mathrm{d}_{\partial\Omega}$ is a unitary extension of the normal field that coincides with $\mathbf{n} \circ p_{\partial\Omega}$ in the tubular neighborhood $\mathcal{T}_{\partial\Omega}$.

As a consequence, the flow Φ_t of the vector field $\boldsymbol{X}_{\boldsymbol{\Theta}}$ defined in $\mathcal{T}_{\partial\Omega}$ by

$$
\boldsymbol{X}_{\boldsymbol{\Theta}}(x) = h(x)\,\mathbf{n}(p_{\partial\Omega}(x)) \tag{2.1}
$$

defines a path $t \in [0,1] \mapsto \Omega_t$ within domains connecting $\Omega_0 = \Omega$ to $\Omega_1 = \Omega_{per}$. However, this flow does not in general preserve the volume of the domain even if the original and perturbed domains share the same volume. Therefore, an alternative field $\boldsymbol{Y}_{\boldsymbol{\Theta}}$ defined in tubular coordinates (M, h) by

$$
\boldsymbol{Y}_{\boldsymbol{\Theta}}(M, h) = -\frac{\displaystyle\int_0^{\mathrm{d}_{\partial\Omega}(\boldsymbol{\Theta} \circ \boldsymbol{R}_{\boldsymbol{\Theta}}^{-1}(M))} \det \mathrm{D}T_{\partial\Omega}(M, s)\,\mathrm{d}s}{\det \mathrm{D}T_{\partial\Omega}(M, h)}\,\mathbf{n}(M) \tag{2.2}
$$

was build in [9, Subsection 2.1] in order to deal with volume constraint. Indeed, the divergence of $\boldsymbol{Y}_{\boldsymbol{\Theta}}$ cancels in the tubular neighborhood $\mathcal{T}_{\partial\Omega}$ and the flow of $\boldsymbol{Y}_{\boldsymbol{\Theta}}$ preserves the measure of Ω for t in $[0,1]$.

Let us remark that the field $\boldsymbol{X}_\Theta$ involves the normal field $\mathbf{n}$ to $\partial\Omega$ and that $\boldsymbol{Y}_\Theta$ involves first order derivatives of $\mathbf{n}$. Hence their flow is a $\mathcal{C}^{2,\alpha}$-diffeomorphism that is to say let stable the admissible class of domain under additional regularity assumptions. Let us assume from now on that the initial shape is in fact $\mathcal{C}^{4,\alpha}$. For time $t \in [0,1]$, the flow Φ_t of both fields $\boldsymbol{X}_\Theta$ and $\boldsymbol{Y}_\Theta$ remains a perturbation of the identity: there is a constant C such that, for all $t \in [0,1]$:

$$\|\Phi_t - \mathbf{I}\|_{2,\alpha} \leq C\|\boldsymbol{\Theta} - \mathbf{I}\|_{2,\alpha}. \tag{2.3}$$

Noting that the outer normal field $\mathbf{n}_t$ to Ω_t can be written as

$$\mathbf{n}_t(y) = \frac{\nabla(d_{\partial\Omega} \circ \Phi_t^{-1})}{\|\nabla(d_{\partial\Omega} \circ \Phi_t^{-1})\|}(y), \ \forall y \in \partial\Omega,$$

the following estimates are a consequence of the Faà De Bruno's formula of successive derivatives of a composition:

Lemma 2.1. *Let Ω be a bounded open set of $\mathbb{R}^d$ with a $C^{4,\alpha}$ boundary. There is a constant $C > 0$ depending on Ω such that*

- *the surface jacobian $J(t) := \det \mathrm{D}\Phi_t / \|({}^t\mathrm{D}\Phi_t^{-1})\mathbf{n}\|$ satisfies*

$$\|J(t,.) - 1\|_{\mathcal{C}^1(\partial\Omega)} \leq C\|\Phi_t - \mathbf{I}\|_{2,\alpha}, \ \forall t \in [0,1]; \tag{2.4}$$

- *the normal field $\mathbf{n}_t$ to $\partial\Omega_t$ satisfies*

$$\|\mathbf{n}_t(\Phi_t(.)) - \mathbf{n}(.)\|_{\mathcal{C}^1(\partial\Omega)} \leq C\|\Phi_t - \mathbf{I}\|_{2,\alpha}, \ \forall t \in [0,1]. \tag{2.5}$$

- *Set $m_{\boldsymbol{V}} := \boldsymbol{V} \cdot \tilde{\mathbf{n}}$ for $\boldsymbol{V} = \boldsymbol{X}_\Theta$ or $\boldsymbol{Y}_\Theta$, then for all $t \in [0,1]$:*

$$\|m_{\boldsymbol{V}}(\Phi_t(.)) - m_{\boldsymbol{V}}\|_{\mathrm{L}^2(\partial\Omega)} \leq C \|m_{\boldsymbol{V}}\|_{\mathrm{L}^2(\partial\Omega)} \|\Phi_t - I\|_{2,\alpha}, \tag{2.6}$$

$$\|m_{\boldsymbol{V}}(\Phi_t(.)) - m_{\boldsymbol{V}}\|_{\mathrm{H}^{1/2}(\partial\Omega)} \leq C \|m_{\boldsymbol{V}}\|_{\mathrm{H}^{1/2}(\partial\Omega)} \|\Phi_t - I\|_{2,\alpha}. \tag{2.7}$$

Remark 2.2. *The fact that we have to impose a $\mathcal{C}^{4,\alpha}$ regularity of the boundary of the initial shape (whereas we can work with shapes with a $\mathcal{C}^{2,1}$ boundary in order to have the twice differentiability with respect to the domain) comes to the fact that we consider normal perturbations (so we loose one rank of regularity) which are divergence free (which imposes the lost of an additional derivative by construction): see [9, Section 2.1].*

2.3 The general notations and the functional

The drag functional. Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ (with $d = 2$ or $d = 3$) containing a Newtonian and incompressible fluid with coefficient of kinematic viscosity $\nu > 0$. We assume that Ω is smooth (at least with a $\mathcal{C}^{2,1}$ boundary). We also assume that ν is constant. Let $\delta > 0$ fixed (large). We define the set of admissible shapes by

$$\mathcal{O}_\delta := \big\{ \omega \subset \Omega \text{ open set with a } \mathcal{C}^{2,1} \text{ boundary such that } \mathrm{d}(x, \partial\Omega) > \delta, \forall x \in \omega$$
$$\text{and such that } \Omega \backslash \overline{\omega} \text{ is connected} \big\}.$$

4

Let us consider $\boldsymbol{g} \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, with $\alpha \in (0,1)$, satisfying the compatibility condition $\displaystyle\int_{\partial\Omega} \boldsymbol{g} \cdot \mathbf{n} = 0$ and the unique solution $(\boldsymbol{u}, p) \in \mathcal{C}^{2,\alpha}(\overline{\Omega\backslash\overline{\omega}}) \times \left[\mathcal{C}^{1,\alpha}(\overline{\Omega\backslash\overline{\omega}}) \cap \mathrm{L}_0^2(\Omega\backslash\overline{\omega}) \right]$ of

$$
\begin{cases}
-\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{0} & \text{in } \Omega\backslash\overline{\omega} \\
\operatorname{div}\boldsymbol{u} &= 0 & \text{in } \Omega\backslash\overline{\omega} \\
\boldsymbol{u} &= \boldsymbol{g} & \text{on } \partial\Omega \\
\boldsymbol{u} &= \boldsymbol{0} & \text{on } \partial\omega.
\end{cases}
$$

The existence and uniqueness of the solution of such a problem is classical. We refer for example to the book of Galdi [14, Theorems IV.7.1 and IV.7.2]. The energy dissipated by the fluid is given by

$$
J(\omega) := \frac{1}{2} \int_{\Omega\backslash\overline{\omega}} \nu \left| \mathcal{D}(\boldsymbol{u}) \right|^2,
$$

where

$$
\mathcal{D}(\boldsymbol{u}) := \left(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u} \right) = \left(\partial_i u_j + \partial_j u_i \right)_{i,j}, \qquad i,j = 1, \dots, d.
$$

The drag minimization problem is to minimize J over all subdomains ω of Ω with a given measure M:

$$
\omega^* := \operatorname{argmin} \left\{ J(\omega);\ \omega \in \mathcal{O}_\delta \text{ such that } \mathcal{L}^d(\omega) = M \right\},
$$

where $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$.

We also define the following stress tensor:

$$
\sigma(\boldsymbol{u}, p) := \nu \left(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u} \right) - p\,\mathrm{I}.
$$

Admissible deformations. Let $\omega \in \mathcal{O}_\delta$. Let us define some admissible deformations of the domain ω we will use in this paper. Since we want to perturb only ω (and not Ω which is fixed), we define Ω_δ an open set with a $\mathcal{C}^\infty$ boundary and such that

$$
\left\{ x \in \Omega;\ \mathrm{d}(x, \partial\Omega) > \delta/2 \right\} \subset \Omega_\delta \subset \left\{ x \in \Omega;\ \mathrm{d}(x, \partial\Omega) > \delta/3 \right\}.
$$

We then consider a diffeomorphism

$$
\boldsymbol{\Theta} \in \mathfrak{U} := \left\{ \boldsymbol{\theta} \in \mathcal{C}^{2,1}(\mathbb{R}^d),\ \boldsymbol{\theta} \equiv \mathrm{I} \text{ in } \mathbb{R}^d\backslash\Omega_\delta \right\}.
$$

The principle of Hadamard's approach is to consider the flow $\Phi_{\boldsymbol{\Theta},t}$ of an adequate autonomous vector field

$$
\boldsymbol{V}_{\boldsymbol{\Theta}} \in \boldsymbol{U} := \left\{ \boldsymbol{\theta} \in \mathcal{C}^{2,1}(\mathbb{R}^d);\ \operatorname{Supp} \boldsymbol{\theta} \subset \overline{\Omega_\delta} \right\},
$$

i.e. the solution of

$$
\begin{cases}
\partial_t \Phi &= \boldsymbol{V}_{\boldsymbol{\Theta}}(\Phi) \\
\Phi(0, x) &= x.
\end{cases}
$$

Notice that $\boldsymbol{V}_{\boldsymbol{\Theta}}$ represents one of the previous fields $\boldsymbol{X}_{\boldsymbol{\Theta}}$ or $\boldsymbol{Y}_{\boldsymbol{\Theta}}$ defined in Section 2.2. It defines a regular path $(\omega_t := \Phi_{\boldsymbol{\Theta},t}(\omega))_{t \in [0,1]}$ in $\mathcal{O}_\delta$. For the rest of the paper, we use a subscript "$_t$" to indicate that the quantity is defined on the time t dependent domain. For instance, $\mathbf{n}_t$ is the external unit normal of $\Omega\backslash\overline{\omega_t}$. Moreover, in order to simplify the notations, we omit the dependency with respect to $\boldsymbol{\Theta}$: in particular, we use the notation Φ_t instead of $\Phi_{\boldsymbol{\Theta},t}$.

Let us now construct some vector fields whose flows connects the original shape ω^* to the perturbed domain $\omega_1 = \boldsymbol{\Theta}(\omega)$.

3 Computation of the shape Hessian

In this section, we prove that the shape Hessian of the functional J can be coercive for a critical shape $\omega^* \in \mathcal{O}_\delta$ with a $\mathcal{C}^{4,\alpha}$ boundary in the $\mathrm{H}^{1/2}(\partial\omega^*)$ norm. We assume the existence of such a critical shape. We first define, for $t \in [0,1]$,

$$j(t) := J(\omega_t) := \frac{1}{2} \int_{\Omega\setminus\overline{\omega_t}} \nu\, |\mathcal{D}(\boldsymbol{u}_t)|^2 ,$$

where $(\boldsymbol{u}_t, p_t) \in \boldsymbol{\mathcal{C}}^{2,\alpha}(\overline{\Omega\setminus\omega_t}) \times \left[\mathcal{C}^{1,\alpha}(\overline{\Omega\setminus\omega_t}) \cap \mathrm{L}_0^2(\Omega\setminus\overline{\omega_t})\right]$ is the solution of the following perturbed problem:

$$\begin{cases} -\nu\Delta\boldsymbol{u}_t + \nabla p_t &=& \boldsymbol{0} & \text{in } \Omega\setminus\overline{\omega_t} \\ \operatorname{div}\boldsymbol{u}_t &=& 0 & \text{in } \Omega\setminus\overline{\omega_t} \\ \boldsymbol{u}_t &=& \boldsymbol{g} & \text{on } \partial\Omega \\ \boldsymbol{u}_t &=& \boldsymbol{0} & \text{on } \partial\omega_t, \end{cases} \tag{3.1}$$

with $\omega_t := \Phi_t(\omega^*)$. The existence of the second order shape derivatives is proved in some previous works (see for example [20, 3, 8] for the Stokes case and [5, 7] for the Navier-Stokes case).

Let $\boldsymbol{V} \in \boldsymbol{U}$ be divergence free. Classically, the shape derivative $(\boldsymbol{u}'_t, p'_t)$ which belongs to $\boldsymbol{\mathcal{C}}^{2,\alpha}(\overline{\Omega\setminus\omega_t}) \times \left[\mathcal{C}^{1,\alpha}(\overline{\Omega\setminus\omega_t}) \cap \mathrm{L}_0^2(\Omega\setminus\overline{\omega_t})\right]$ is characterized as the solution of the following Stokes problem:

$$\begin{cases} -\nu\Delta\boldsymbol{u}'_t + \nabla p'_t &=& \boldsymbol{0} & \text{in } \Omega\setminus\overline{\omega_t} \\ \operatorname{div}\boldsymbol{u}'_t &=& 0 & \text{in } \Omega\setminus\overline{\omega_t} \\ \boldsymbol{u}'_t &=& \boldsymbol{0} & \text{on } \partial\Omega \\ \boldsymbol{u}'_t &=& -\partial_{\mathbf{n}_t}\boldsymbol{u}_t\,(\boldsymbol{V}\cdot\mathbf{n}_t) & \text{on } \partial\omega_t. \end{cases} \tag{3.2}$$

3.1 Shape gradient of the functional and Euler-Lagrange equation

Simon proves in [20, Theorem 3] that

$$\mathrm{D}J(\omega^*)\cdot\boldsymbol{V} = -\frac{1}{2}\int_{\partial\omega^*}\nu\,|\partial_{\mathbf{n}}\boldsymbol{u}|^2\,(\boldsymbol{V}\cdot\mathbf{n}).$$

Since we work under the constraint of constant volume and since ω^* is a critical point, there exists $\Lambda_0 \in \mathbb{R}$ such that $\mathrm{D}J(\omega^*)\cdot\boldsymbol{V} + \Lambda_0\,\mathrm{D}\mathcal{V}(\omega^*)\cdot\boldsymbol{V} = 0$ for any perturbation $\boldsymbol{V}$ (where $\mathcal{V}(\omega^*)$ is the volume of ω^*). Hence, for all $\boldsymbol{V} \in \boldsymbol{U}$,

$$-\frac{1}{2}\int_{\partial\omega^*}\nu\,|\partial_{\mathbf{n}}\boldsymbol{u}|^2\,(\boldsymbol{V}\cdot\mathbf{n}) + \Lambda_0\int_{\partial\omega^*}\boldsymbol{V}\cdot\mathbf{n} = 0,$$

and then we obtain the Euler-Lagrange equation satisfied at the critical shape: there exists $\Lambda \in \mathbb{R}$ such that

$$|\partial_{\mathbf{n}}\boldsymbol{u}|^2 = \frac{2}{\nu}\Lambda_0 =: \Lambda. \tag{3.3}$$

Notice that this result is also proved in [20, Theorem 7] in a different way.

3.2 Shape Hessian of the functional

Let T be a fixed small positive real number. Simon proves in [20, Theorem 4] that, for $t \in [0,T)$,

$$j'(t) = -\frac{1}{2}\int_{\partial\omega_t}\nu\,|\mathcal{D}(\boldsymbol{u}_t)|^2\,(\boldsymbol{V}\cdot\mathbf{n}).$$

In [20, Theorem 8], the second variation $j''(0)$ is computed. However, we shall use a simplified expression of $j''(0)$ which, in particular, does not depend on the second order shape derivative of the state $\boldsymbol{u}$. Let us obtain this new expression.

Since $\boldsymbol{V} = \boldsymbol{0}$ on $\partial\Omega$, we use Green's formula to get

$$j'(t) = -\frac{1}{2}\int_{\Omega\setminus\overline{\omega_t}} \nu\,\mathrm{div}\left(|\mathcal{D}(\boldsymbol{u}_t)|^2\,\boldsymbol{V}\right).$$

Thus, we use Hadamard's formula (see [15, Corollary 5.2.5]) to derive this function. For $t \in [0,T)$, it holds

$$
\begin{aligned}
j''(t) &= -\int_{\Omega\setminus\overline{\omega_t}} \nu\,\mathrm{div}\left((\mathcal{D}(\boldsymbol{u}_t):\mathcal{D}(\boldsymbol{u}_t'))\boldsymbol{V}\right) - \frac{1}{2}\int_{\partial\omega_t} \nu\,\mathrm{div}\left(|\mathcal{D}(\boldsymbol{u}_t)|^2\,\boldsymbol{V}\right)(\boldsymbol{V}\cdot\mathbf{n}_t) \\
&= -\frac{1}{2}\int_{\partial\omega_t} \nu\left(2(\mathcal{D}(\boldsymbol{u}_t):\mathcal{D}(\boldsymbol{u}_t')) + \nabla(|\mathcal{D}(\boldsymbol{u}_t)|^2)\cdot\boldsymbol{V} + |\mathcal{D}(\boldsymbol{u}_t)|^2\,\mathrm{div}\,\boldsymbol{V}\right)(\boldsymbol{V}\cdot\mathbf{n}_t) \\
&= -\frac{1}{2}\int_{\partial\omega_t} \nu\left(2(\mathcal{D}(\boldsymbol{u}_t):\mathcal{D}(\boldsymbol{u}_t')) + \nabla(|\mathcal{D}(\boldsymbol{u}_t)|^2)\cdot\boldsymbol{V}\right)(\boldsymbol{V}\cdot\mathbf{n}_t), \qquad (3.4)
\end{aligned}
$$

since we work with a divergence free vector field $\boldsymbol{V}$. Moreover,

$$
\begin{aligned}
\int_{\partial\omega_t} \nu\,\mathcal{D}(\boldsymbol{u}_t):\mathcal{D}(\boldsymbol{u}_t')\,(\boldsymbol{V}\cdot\mathbf{n}_t) &= 2\int_{\partial\omega_t} \nu\,\nabla\boldsymbol{u}_t:\mathcal{D}(\boldsymbol{u}_t')(\boldsymbol{V}\cdot\mathbf{n}_t) \\
&= 2\int_{\partial\omega_t} \nabla\boldsymbol{u}_t:(\nu\,\mathcal{D}(\boldsymbol{u}_t') - p_t'\mathrm{I})(\boldsymbol{V}\cdot\mathbf{n}_t).
\end{aligned}
$$

Here we used that the solution $\boldsymbol{u}_t$ of Stokes problem is divergence free in $\Omega\setminus\overline{\omega_t}$ to get

$$\int_{\partial\omega_t} \nabla\boldsymbol{u}_t:(p_t'\,\mathrm{I})(\boldsymbol{V}\cdot\mathbf{n}_t) = \int_{\partial\omega_t} p_t'\,\mathrm{div}\,\boldsymbol{u}_t(\boldsymbol{V}\cdot\mathbf{n}_t) = 0.$$

Moreover, $\boldsymbol{u}_t$ satisfies the Dirichlet boundary condition $\boldsymbol{u}_t = \boldsymbol{0}$ on $\partial\omega_t$, and then it follows that $\nabla\boldsymbol{u}_t = \partial_{\mathbf{n}_t}\boldsymbol{u}_t \otimes \mathbf{n}_t$ on $\partial\omega_t$. Thus,

$$
\begin{aligned}
\int_{\partial\omega_t} \nu\mathcal{D}(\boldsymbol{u}_t):\mathcal{D}(\boldsymbol{u}_t')\,(\boldsymbol{V}\cdot\mathbf{n}_t) &= 2\int_{\partial\omega_t} \nabla\boldsymbol{u}_t:\sigma(\boldsymbol{u}_t',p_t')(\boldsymbol{V}\cdot\mathbf{n}_t) \\
&= 2\int_{\partial\omega_t} \partial_{\mathbf{n}_t}\boldsymbol{u}_t \cdot \left(\sigma(\boldsymbol{u}_t',p_t')\,\mathbf{n}_t\right)(\boldsymbol{V}\cdot\mathbf{n}_t). \qquad (3.5)
\end{aligned}
$$

Then, combining Equalities (3.4) and (3.5),

$$j''(t) = -2\int_{\partial\omega_t} \partial_{\mathbf{n}_t}\boldsymbol{u}_t \cdot (\sigma(\boldsymbol{u}_t',p_t')\mathbf{n}_t)(\boldsymbol{V}\cdot\mathbf{n}_t) - \frac{1}{2}\int_{\partial\omega_t} \nu\nabla(|\mathcal{D}(\boldsymbol{u}_t)|^2)\cdot\boldsymbol{V}(\boldsymbol{V}\cdot\mathbf{n}_t).$$

Finally, since the shape derivative $\boldsymbol{u}_t'$ satisfies (3.2) and in particular $\boldsymbol{u}_t' = -\partial_{\mathbf{n}_t}\boldsymbol{u}_t(\boldsymbol{V}\cdot\mathbf{n}_t)$ on $\partial\omega_t$,

$$j''(t) = 2\int_{\partial\omega_t} \boldsymbol{u}_t' \cdot (\sigma(\boldsymbol{u}_t',p_t')\mathbf{n}_t) - \frac{1}{2}\int_{\partial\omega_t} \nu\nabla(|\mathcal{D}(\boldsymbol{u}_t)|^2)\cdot\boldsymbol{V}(\boldsymbol{V}\cdot\mathbf{n}_t). \qquad (3.6)$$

3.3 At a critical shape

We recall that ω^* is a critical shape. Now, we study the expression of $j''(0)$ and then use the fact that ω^* is a critical shape. The previous characterization of $j''(t)$ gives, for $t = 0$, the expression

$$j''(0) = \underbrace{2\int_{\partial\omega^*} \boldsymbol{u}' \cdot (\sigma(\boldsymbol{u}',p')\mathbf{n})}_{=:J_1} - \underbrace{\frac{1}{2}\int_{\partial\omega^*} \nu\nabla(|\mathcal{D}(\boldsymbol{u})|^2)\cdot\boldsymbol{V}(\boldsymbol{V}\cdot\mathbf{n})}_{=:J_2}. \qquad (3.7)$$

that we split into two terms we will study separately.

Study of J_1. We first introduce the Steklov-Poincaré operator of Displacement-to-Traction also called *Dirichlet-to-Neumann* operator and prove that this operator is positive.

Lemma 3.1. *We consider Ω and ω two Lipschitz open sets of $\mathbb{R}^d$ ($d = 2, 3$) such that $\omega \subset\subset \Omega$ and $\Omega\backslash\overline{\omega}$ is connected. For $\boldsymbol{h} \in \mathbf{H}^{1/2}(\partial\omega)$, let us define the following Steklov-Poincaré operator:*

$$
\begin{aligned}
D_N: \quad \mathbf{H}^{1/2}(\partial\omega) &\longrightarrow \mathbf{H}^{-1/2}(\partial\omega)\\
\boldsymbol{h} &\longmapsto \sigma(\boldsymbol{U}, P)\mathbf{n},
\end{aligned}
$$

where $(\boldsymbol{U}, P) \in \mathbf{H}^1(\Omega\backslash\overline{\omega}) \times \mathrm{L}^2(\Omega\backslash\overline{\omega})$ solves

$$
\begin{cases}
-\operatorname{div}(\sigma(\boldsymbol{U}, P)) &= \boldsymbol{0} &\text{in } \Omega\backslash\overline{\omega}\\
\operatorname{div}\boldsymbol{U} &= 0 &\text{in } \Omega\backslash\overline{\omega}\\
\boldsymbol{U} &= \boldsymbol{0} &\text{on } \partial\Omega\\
\boldsymbol{U} &= \boldsymbol{h} &\text{on } \partial\omega.
\end{cases}
\tag{3.8}
$$

Then, there exists a constant c depending of Ω and ω such that

$$
\langle D_N(\boldsymbol{h}), \boldsymbol{h}\rangle_{\mathbf{H}^{-1/2}(\partial\omega),\mathbf{H}^{1/2}(\partial\omega)} \geq c\,\|\boldsymbol{h}\|^2_{\mathbf{H}^{1/2}(\partial\omega)}.
\tag{3.9}
$$

Proof. Using $\boldsymbol{U}$ as a test function in Problem (3.8), we obtain

$$
\langle D_N(\boldsymbol{h}), \boldsymbol{h}\rangle_{\mathbf{H}^{-1/2}(\partial\omega),\mathbf{H}^{1/2}(\partial\omega)} = \frac{1}{2}\nu\,\|\mathcal{D}(\boldsymbol{U})\|^2_{\mathbf{L}^2(\Omega\backslash\overline{\omega})}.
$$

Then, since $\boldsymbol{U} = \boldsymbol{0}$ on $\partial\Omega$, Korn's inequality (see for example [16, eq. (2.14) page 19]) leads

$$
\langle D_N(\boldsymbol{h}), \boldsymbol{h}\rangle_{\mathbf{H}^{-1/2}(\partial\omega),\mathbf{H}^{1/2}(\partial\omega)} \geq c\,\|\boldsymbol{U}\|^2_{\mathbf{H}^1(\Omega\backslash\overline{\omega})} \geq c\,\|\boldsymbol{h}\|^2_{\mathbf{H}^{1/2}(\partial\omega)}.
$$

$\square$

Consider Problem (3.2) (for $t = 0$) solved by the shape derivative $(\boldsymbol{u}', p')$. Then, noticing that, since $\operatorname{div}\boldsymbol{u}' = 0$,

$$
-\nu\Delta\boldsymbol{u}' + \nabla p' = -\operatorname{div}(\nu\mathcal{D}(\boldsymbol{u}')) + \nabla p' = -\operatorname{div}(\sigma(\boldsymbol{u}', p')),
$$

and that $\boldsymbol{u}' = -\partial_{\mathbf{n}}\boldsymbol{u}\,(\boldsymbol{V}\cdot\mathbf{n})$ on $\partial\omega^*$, we get

$$
\int_{\partial\omega^*} \boldsymbol{u}'\cdot(\sigma(\boldsymbol{u}', p')\mathbf{n}) = \int_{\partial\omega^*} \partial_{\mathbf{n}}\boldsymbol{u}(\boldsymbol{V}\cdot\mathbf{n})\,D_N(\partial_{\mathbf{n}}\boldsymbol{u}(\boldsymbol{V}\cdot\mathbf{n})).
$$

Hence, there exists a constant $c > 0$ such that

$$
J_1 \geq c\,\|\partial_{\mathbf{n}}\boldsymbol{u}\,(\boldsymbol{V}\cdot\mathbf{n})\|^2_{\mathbf{H}^{1/2}(\partial\omega^*)} = c\,\Lambda\,\|\boldsymbol{V}\cdot\mathbf{n}\|^2_{\mathbf{H}^{1/2}(\partial\omega^*)}.
\tag{3.10}
$$

We used the coercivity of the operator D_N ((3.9) in Lemma 3.1) and the Euler-Lagrange equation $|\partial_{\mathbf{n}}\boldsymbol{u}|^2 = \Lambda$ according to (3.3)

Second term J_2. Let us now study the second term of the characterization of $j''(0)$ given by (3.7). We recall that, according to subsection 2.2, only normal perturbation directions are considered. Thus, we focus on

$$
\int_{\partial\omega^*} \nu\nabla(|\mathcal{D}(\boldsymbol{u})|^2)\cdot\mathbf{n}\,(\boldsymbol{V}\cdot\mathbf{n})^2.
$$

We have to compute $\nu\nabla(|\mathcal{D}(\boldsymbol{u})|^2)\cdot\mathbf{n}$. This is the object of next Lemma.

8

Lemma 3.2. *It holds*

$$\nu\nabla\left(|\mathcal{D}(\boldsymbol{u})|^2\right)\cdot\mathbf{n} = 4\left(\nu(d-1)\mathrm{H}\Lambda + \nu\,{}^t\mathfrak{b}\,\mathfrak{F}\,\mathfrak{b} + \nabla_\tau p\cdot\mathfrak{b}\right) \qquad on\ \partial\omega^*, \qquad (3.11)$$

where

- $\mathfrak{b}$ *is the projection of $\partial_{\mathbf{n}}\boldsymbol{u}$ on the tangential space of $\partial\omega^*$;*

- $\mathfrak{F}$ *is the second fundamental form of the surface $\partial\omega^*$;*

- ∇_τ *denotes the tangential gradient.*

Proof. By using local coordinates as in [11] or [10, 9], we can assume that ω^* is locally above the graph of a function $f : (-\varepsilon,\varepsilon)^{d-1} \to \mathbb{R}$ (of class $\mathcal{C}^{4,\alpha}$) which is such that $f(0) = \partial_i f(0) = 0$ for $i = 1,\ldots,d-1$.

For all $\overline{x} \in (-\varepsilon,\varepsilon)^{d-1}$, noticing $\mathbf{n} = (n_1,\ldots,n_d)$,

$$\nabla\left(|\mathcal{D}(\boldsymbol{u})(\overline{x},f(\overline{x}))|^2\right)\cdot\mathbf{n}(\overline{x},f(\overline{x})) = \nabla\left(\sum_{i,j=1}^{d}(\partial_i u_j(\overline{x},f(\overline{x})) + \partial_j u_i(\overline{x},f(\overline{x})))^2\right)\cdot\mathbf{n}(\overline{x},f(\overline{x}))$$

$$= \sum_{k=1}^{d}\partial_k\left(\sum_{i,j=1}^{d}(\partial_i u_j(\overline{x},f(\overline{x})) + \partial_j u_i(\overline{x},f(\overline{x})))^2\right)n_k(\overline{x},f(\overline{x}))$$

$$= 2\sum_{i,j,k=1}^{d}(\partial_{ki}^2 u_j(\overline{x},f(\overline{x})) + \partial_{kj}^2 u_i(\overline{x},f(\overline{x})))(\partial_i u_j(\overline{x},f(\overline{x})) + \partial_j u_i(\overline{x},f(\overline{x})))n_k(\overline{x},f(\overline{x})).$$

Then, for $\overline{x} = 0$, since $\mathbf{n}(0,f(0)) = (0,\ldots,0,1)$ in our system of coordinates,

$$\nabla\left(|\mathcal{D}(\boldsymbol{u})(0,f(0))|^2\right)\cdot\mathbf{n}(0,f(0))$$

$$= 2\sum_{i,j=1}^{d}(\partial_{di}^2 u_j(0,f(0)) + \partial_{dj}^2 u_i(0,f(0)))(\partial_i u_j(0,f(0)) + \partial_j u_i(0,f(0))).$$

Moreover, since $\boldsymbol{u} = \mathbf{0}$ on $\partial\omega^*$, we have $\partial_i u_j(0,f(0)) = 0$ for all $i = 1,\ldots,d-1$ and $j = 1,\ldots,d$. In addition, since $\mathrm{div}\,\boldsymbol{u} = 0$ in $\Omega\backslash\overline{\omega}$, it follows that $\partial_d u_d(0,f(0)) = 0$. Then,

$$\nu\nabla\left(|\mathcal{D}(\boldsymbol{u})(0,f(0))|^2\right)\cdot\mathbf{n}(0,f(0)) = 4\sum_{j=1}^{d-1}\nu\left(\partial_{dd}^2 u_j(0,f(0)) + \partial_{dj}^2 u_d(0,f(0))\right)\partial_d u_j(0,f(0)).$$

$$(3.12)$$

Let us now interpret the boundary condition on $\partial\omega^*$, that is

$$\boldsymbol{u}(\overline{x},f(\overline{x})) = \mathbf{0} \qquad \text{for } \overline{x} \in (-\varepsilon,\varepsilon)^{d-1}.$$

By differentiation, for $i,j = 1,\ldots,d-1$ and $k = 1,\ldots,d$,

$$\partial_j u_k(\overline{x},f(\overline{x})) + \partial_j f(\overline{x})\partial_d u_k(\overline{x},f(\overline{x})) = 0$$

and

$$\partial_{ij}^2 u_k(\overline{x},f(\overline{x})) + \partial_i f(\overline{x})\partial_{dj}^2 u_k(\overline{x},f(\overline{x})) + \partial_{ij}^2 f(\overline{x})\partial_d u_k(\overline{x},f(\overline{x}))$$
$$+ \partial_j f(\overline{x})\left(\partial_{id}^2 u_k(\overline{x},f(\overline{x})) + \partial_i f(\overline{x})\partial_{dd}^2 u_k(\overline{x},f(\overline{x}))\right) = 0.$$

9

For $\overline{x} = 0$, since $f(0) = \partial_i f(0) = 0$ for $i = 1, \ldots, d-1$,

$$\partial^2_{ij} u_k(0, f(0)) + \partial^2_{ij} f(0) \partial_d u_k(0, f(0)) = 0 \qquad \forall i, j = 1, \ldots, d-1, \quad \forall k = 1, \ldots, d. \quad (3.13)$$

Notice that $-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{0}$ in $\Omega \backslash \overline{\omega}$, *i.e.*, since $\operatorname{div} \boldsymbol{u} = 0$ in $\Omega \backslash \overline{\omega}$, $-\nu \operatorname{div}(\mathcal{D}(\boldsymbol{u})) + \nabla p = 0$ in $\Omega \backslash \overline{\omega}$. Hence, for $j = 1, \ldots, d$ and any $\overline{x} \in (-\varepsilon, \varepsilon)^{d-1}$,

$$\nu \big(\partial^2_{dd} u_j(\overline{x}, f(\overline{x})) + \partial^2_{dj} u_d(\overline{x}, f(\overline{x})) \big) = -\sum_{i=1}^{d-1} \big[\nu \big(\partial^2_{ii} u_j(\overline{x}, f(\overline{x})) + \partial^2_{ij} u_i(\overline{x}, f(\overline{x})) \big) + \partial_j p(\overline{x}, f(\overline{x})) \big] .$$

$$(3.14)$$

In Equality (3.13) with $j = i$, we sum on i to get, since $\displaystyle\sum_{i=1}^{d-1} \partial^2_{ii} f(0) = (d-1)\mathrm{H}(0)$, where H is the mean curvature of $\partial \omega^*$ (see for example the comment after Definition 5.4.7 in [15]), that for all $k = 1, \ldots, d$

$$\sum_{i=1}^{d-1} \partial^2_{ii} u_k(0, f(0)) + (d-1)\mathrm{H}(0)\partial_d u_k(0, f(0)) = 0. \qquad (3.15)$$

Hence, using (3.13) and (3.15), Equality (3.14) writes as follows for $j = 1, \ldots, d-1$:

$$\nu \left(\partial^2_{dd} u_j(0, f(0)) + \partial^2_{dj} u_d(0, f(0)) \right)$$

$$= \nu (d-1)\mathrm{H}(0) \, \partial_d u_j(0, f(0)) + \nu \sum_{i=1}^{d-1} \partial^2_{ij} f(0)\partial_d u_i(0, f(0)) + \partial_j p(0, f(0)).$$

Therefore, Equality (3.12) writes in the following form:

$$\nu \nabla \left(|\mathcal{D}(\boldsymbol{u})(0, f(0))|^2 \right) \cdot \mathbf{n}(0, f(0)) =$$

$$4 \sum_{j=1}^{d-1} \left[\left(\nu(d-1)\mathrm{H}(0) \, (\partial_d u_j(0, f(0)))^2 + \nu \sum_{i=1}^{d-1} \partial^2_{ij} f(0)\partial_d u_i(0, f(0))\partial_d u_j(0, f(0)) \right. \right.$$

$$\left. \left. + \partial_j p(0, f(0))\partial_d u_j(0, f(0)) \right) \right].$$

Hence, since $|\partial_d \boldsymbol{u}|^2 = \Lambda$ (see (3.3)), Lemma 3.2 follows. $\qquad \square$

Conclusion. Finally, gathering (3.7), (3.10) and (3.11), we obtain

$$\mathrm{D}^2 J(\omega^*) \cdot \boldsymbol{V} \cdot \boldsymbol{V} \geq c\Lambda \, \|\boldsymbol{V} \cdot \mathbf{n}\|^2_{\mathrm{H}^{1/2}(\partial \omega^*)} - 2 \int_{\partial \omega^*} \big(\nu(d-1)\mathrm{H}\Lambda + \nu \, {}^t\mathfrak{b}\,\mathfrak{F}\,\mathfrak{b} + \nabla_\tau p \cdot \mathfrak{b} \big) (\boldsymbol{V} \cdot \mathbf{n})^2.$$

Hence, using the regularity of $\partial \omega^*$ and of the solution $\boldsymbol{u}$, the quantity

$$\nu(d-1)\mathrm{H}\Lambda + \nu \, {}^t\mathfrak{b}\,\mathfrak{F}\,\mathfrak{b} + \nabla_\tau p \cdot \mathfrak{b}$$

belongs to $\mathrm{L}^\infty(\partial \omega^*)$ and the second term behaves like $\|\boldsymbol{V} \cdot \mathbf{n}\|^2_{\mathrm{L}^2(\partial \omega^*)}$. Hence, a natural assumption is that the shape Hessian is coercive in the $\mathrm{H}^{1/2}(\partial \omega^*)$ sense. This is the case if

$$\nu(d-1)\mathrm{H}\Lambda + \nu \, {}^t\mathfrak{b}\,\mathfrak{F}\,\mathfrak{b} + \nabla_\tau p \cdot \mathfrak{b} < 0 \text{ on } \partial \omega^*. \qquad (3.16)$$

Note that this condition couples geometrical effect with the solution itself. In practice, the condition (3.16) cannot easily be tested theoretically since it couples the curvature of the object with derivatives of the flow. It might be tested numerically, however it will requires a curved mesh for the surface and the fluid domain and high order elements to catch the desired effects of curvature and the derivatives of the couple $(\boldsymbol{u}, p)$. Such a precise computation requires specific numerical attention.

Statement of the result We have proved the following theorem:

Theorem 3.3. *Let us assume that Estimate* (3.16) *is satisfied. Then, there exists a constant $c > 0$ such that, for all $\boldsymbol{V} \in \boldsymbol{U}$ with* $\mathrm{div}\, \boldsymbol{V} = 0$,

$$j''(0) = \mathrm{D}^2 J(\omega^*) \cdot \boldsymbol{V} \cdot \boldsymbol{V} \geq c \, \|\boldsymbol{V} \cdot \mathbf{n}\|^2_{\mathrm{H}^{1/2}(\partial \omega^*)}.$$

4 Stability of the drag minimization problem

4.1 The results

We recall that $\omega^* \in \mathcal{O}_\delta$ is assumed to have a $\mathcal{C}^{4,\alpha}$ boundary ($\alpha \in (0,1)$). Moreover, in this section, the shape Hessian is assumed to be coercive at ω^* in the following sense: for any $\boldsymbol{V}$ in the tangent space defined by the constraints that is with

$$\int_{\partial \omega^*} \boldsymbol{V} \cdot \mathbf{n} = 0,$$

it holds

$$j''(0) = \mathrm{D}^2 J(\omega^*) \cdot \boldsymbol{V} \cdot \boldsymbol{V} \geq c \, \|\boldsymbol{V} \cdot \mathbf{n}\|^2_{\mathrm{H}^{1/2}(\partial \omega^*)}. \tag{4.1}$$

The main result of this section states the stability of the drag minimization problem:

Theorem 4.1. *If ω^* is a critical shape for J where* (4.1) *holds, there exists $\eta > 0$ such that, for all $\boldsymbol{\Theta} \in \mathfrak{U}$ with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta$, $|\boldsymbol{\Theta}(\omega^*)| = |\omega^*|$ and $\boldsymbol{\Theta} \neq \mathbf{I}$,*

$$J(\boldsymbol{\Theta}(\omega^*)) > J(\omega^*).$$

Consider the admissible vector field $\boldsymbol{Y}_\Theta \in \boldsymbol{U}$ defined in subsection 2.2. By assumption

$$j''(0) \geq C \, \|\boldsymbol{Y}_\Theta \cdot \mathbf{n}\|^2_{\mathrm{H}^{1/2}(\partial \omega^*)}.$$

By the order two Taylor expansion

$$J(\omega) = j(1) = j(0) + \int_0^1 (1-x) j''(x) \mathrm{d}x = J(\omega^*) + \int_0^1 (1-x) j''(x) \mathrm{d}x,$$

this theorem is a direct consequence of the following theorem (and of the assumption of the $\mathrm{H}^{1/2}$ coercivity of the shape Hessian):

Theorem 4.2. *There exists $\eta_0 > 0$ and a function $w : (0, \eta_0) \to \mathbb{R}$ with $\lim_{r \searrow 0} w(r) = 0$ (which depends only on Ω, ω^* and the data) such that, for all $\eta \in (0, \eta_0)$ and all $\boldsymbol{\Theta} \in \mathfrak{U}$ with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta$ and $|\boldsymbol{\Theta}(\omega^*)| = |\omega^*|$, there exists a divergence free vector field $\boldsymbol{Y}_\Theta \in \boldsymbol{U}$ constructed in* (2.2) *whose the flow Φ_t defines a path $(\omega_t := \Phi_t(\omega^*))_{t \in [0,1]}$ between ω^* and $\boldsymbol{\Theta}(\omega^*)$, such that, for all $t \in [0,1]$, the following estimate holds:*

$$\left| j''(t) - j''(0) \right| \leq w(\eta) \, \|\boldsymbol{Y}_\Theta \cdot \mathbf{n}\|^2_{\mathrm{H}^{1/2}(\partial \omega^*)}.$$

Indeed, there is a non negative η such that $w(\eta) \leq C/2$, so that

$$J(\omega) \geq J(\omega^*) + \frac{C}{4} \|\boldsymbol{Y}_\Theta \cdot \mathbf{n}\|^2_{\mathrm{H}^{1/2}(\partial \omega^*)} > J(\omega^*). \tag{4.2}$$

In the following section, we prove Theorem 4.2 that extends to the Stokes case [9, Theorem 3] (or [10, Theorem 2.1]). We follow the same strategy that the one used in these references.

Let $\boldsymbol{\Theta} \in \mathfrak{U}$ with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}$ small enough. For $t \in [0,1]$, we consider the solution $(\boldsymbol{u}_t, p_t) \in \mathcal{C}^{2,\alpha}(\overline{\Omega \backslash \omega_t}) \times \left[C^{1,\alpha}(\overline{\Omega \backslash \omega_t}) \cap \mathrm{L}_0^2(\Omega \backslash \overline{\omega_t}) \right]$ of the perturbed Stokes problem (3.1). We define

$$\boldsymbol{v}_t := \boldsymbol{u}_t \circ \Phi_t \quad \text{and} \quad q_t := p_t \circ \Phi_t.$$

Hence, using the change of variables $x = \Phi_t y$ in Problem (3.1), we prove that

$$\begin{cases} -\mathrm{div}\,(\nabla \boldsymbol{v}_t A(t)) + \mathrm{div}\,(q_t B(t)) & = & \mathbf{0} & \text{in } \Omega \backslash \overline{\omega^*} \\ \nabla \boldsymbol{v}_t : B(t) & = & 0 & \text{in } \Omega \backslash \overline{\omega^*} \\ \boldsymbol{v}_t & = & \boldsymbol{g} & \text{in } \partial \Omega \\ \boldsymbol{v}_t & = & \mathbf{0} & \text{in } \partial \omega^*, \end{cases} \tag{4.3}$$

with

$$\begin{aligned} Jac_t &:= \det\,(\mathrm{D}\Phi_t) \in \mathcal{C}^{1,1}(\mathbb{R}^d) \\ A(t) &:= Jac_t\,\nu\,(\mathrm{D}\Phi_t)^{-1}\,({}^t\mathrm{D}\Phi_t)^{-1} \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathcal{M}_{d,d}) \\ B(t) &:= Jac_t\,({}^t\mathrm{D}\Phi_t)^{-1} \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathcal{M}_{d,d}). \end{aligned}$$

4.2 Preliminary results: bounds on the state function

Let us first focus on the proof of the following preliminary result:

Proposition 4.3. *There exists a function* $w : [0,1] \to \mathbb{R}$ *with* $\lim\limits_{r \searrow 0} w(r) = 0$ *such that*

$$\begin{aligned} \sup_{t \in [0,1]} \|\boldsymbol{u}_t \circ \Phi_t - \boldsymbol{u}\|_{\mathcal{C}^2(\overline{\Omega \backslash \omega^*})} &\leq w\left(\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}\right) \\ \sup_{t \in [0,1]} \|p_t \circ \Phi_t - p\|_{C^1(\overline{\Omega \backslash \omega^*})} &\leq w\left(\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}\right). \end{aligned}$$

Notice that we can assume the existence of a constant C such that:

$$w(\eta) \geq C\eta. \tag{4.4}$$

since if $w(\eta) < C\eta$, we can redefine w as $w(\eta) = C\eta$ (which tends to 0 with η).

First, let us note that, using a differentiability result similar than the one given in [3, Lemma 3.2], there exist $\eta_2 > 0$ and a constant $c > 0$, depending only on the data, such that, for all $\boldsymbol{\Theta} \in \mathfrak{U}$ with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta_2$,

$$\|\boldsymbol{v}_t - \boldsymbol{u}\|_{\mathcal{C}^{2,\alpha}(\Omega \backslash \overline{\omega^*})} \leq c\,\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} \quad \text{and} \quad \|q_t - p\|_{C^{1,\alpha}(\Omega \backslash \overline{\omega^*})} \leq c\,\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}.$$

Hence, we deduce immediately the following lemma:

Lemma 4.4. *There exists* $\eta_2 > 0$ *such that, for all* $\boldsymbol{\Theta} \in \mathfrak{U}$ *with* $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta_2$, *there exists a constant* C *depending only on the data and on* $\|\boldsymbol{\Theta}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}$ *such that, for all* $t \in [0,1]$,

$$\|\boldsymbol{v}_t\|_{\mathcal{C}^{2,\alpha}(\Omega \backslash \overline{\omega^*})} \leq C \quad \text{and} \quad \|q_t\|_{C^{1,\alpha}(\Omega \backslash \overline{\omega^*})} \leq C.$$

Proof of Proposition 4.3. We use similar arguments than the ones used in [10, Proof of Proposition 4.1]. The main ingredient is the compact embedding of $\mathcal{C}^{2,\alpha}(\Omega\backslash\overline{\omega^*})$ into $\mathcal{C}^2(\overline{\Omega\backslash\overline{\omega^*}})$ and of $\mathcal{C}^{1,\alpha}(\Omega\backslash\overline{\omega^*})$ into $\mathcal{C}^1(\overline{\Omega\backslash\overline{\omega^*}})$. Let $\eta_2 > 0$ given by Lemma 4.4. Define, for $\eta \in (0, \eta_2)$,

$$w(\eta) := \sup_{\substack{\boldsymbol{\Theta}\in\mathfrak{U},\,\|\boldsymbol{\Theta}-\mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}<\eta \\ t\in[0,1]}} \left(\|\boldsymbol{v}_t - \boldsymbol{u}\|_{\mathcal{C}^2(\overline{\Omega\backslash\overline{\omega^*}})} + \|q_t - p\|_{\mathcal{C}^1(\overline{\Omega\backslash\overline{\omega^*}})} \right).$$

Lemma 4.4 guarantees that this quantity is well defined. Since the estimate of Proposition 4.3 is obvious, it remains to prove that $\lim_{r\searrow 0} w(r) = 0$.

Let us proceed by contradiction assuming that $\lim_{r\searrow 0} w(r) \neq 0$. Then, there exists a real $a > 0$ and some sequences $(t_n)_{n\in\mathbb{N}} \subset [0,1]$, $(\boldsymbol{\Theta}_n)_{n\in\mathbb{N}} \subset \mathfrak{U}$ with $\|\boldsymbol{\Theta}_n - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta_2$ for all $n \in \mathbb{N}$, $(\omega_n := \Phi_{\boldsymbol{\Theta}_n, t_n}(\omega^*))_{n\in\mathbb{N}}$, $(\boldsymbol{v}_n := \boldsymbol{u}_n \circ \Phi_{\boldsymbol{\Theta}_n, t_n})_{n\in\mathbb{N}}$ and $(q_n := p_n \circ \Phi_{\boldsymbol{\Theta}_n, t_n})_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} \|\boldsymbol{\Theta}_n - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} &\leq \tfrac{1}{n} \\ \|\boldsymbol{v}_n - \boldsymbol{u}\|_{\mathcal{C}^2(\overline{\Omega\backslash\overline{\omega^*}})} + \|q_n - p\|_{\mathcal{C}^1(\overline{\Omega\backslash\overline{\omega^*}})} &\geq a > 0. \end{aligned} \tag{4.5}$$

Using Lemma 4.4, the sequence $(\boldsymbol{v}_n)_n$ is bounded in $\mathcal{C}^{2,\alpha}(\Omega\backslash\overline{\omega^*})$ and $(q_n)_n$ is bounded in $\mathcal{C}^{1,\alpha}(\Omega\backslash\overline{\omega^*})$. Hence, since $\mathcal{C}^{2,\alpha}(\Omega\backslash\overline{\omega^*})$ (respectively $\mathcal{C}^{1,\alpha}(\Omega\backslash\overline{\omega^*})$) is compactly imbedding in $\mathcal{C}^2(\overline{\Omega\backslash\overline{\omega^*}})$ (respectively in $\mathcal{C}^1(\overline{\Omega\backslash\overline{\omega^*}})$), there exist two sub-sequences converging respectively in $\mathcal{C}^2(\overline{\Omega\backslash\overline{\omega^*}})$ to some $\boldsymbol{u}_{lim}$ and in $\mathcal{C}^1(\overline{\Omega\backslash\overline{\omega^*}})$ to some p_{lim}. Moreover, there exists a sub-sequence $(t_n) \subset [0,1]$ which converges to some t_{lim}. Then, notice that

$$\begin{aligned} -\operatorname{div}\left(\nabla\boldsymbol{v}_n A(t_n, \boldsymbol{\Theta}_n)\right) &+ \operatorname{div}\left(q_n B(t_n, \boldsymbol{\Theta}_n)\right) + \nu\Delta\boldsymbol{u}_{lim} - \nabla p_{lim} \\ &= -\operatorname{div}\left(\nabla\boldsymbol{v}_n A(t_n, \boldsymbol{\Theta}_n)\right) + \operatorname{div}\left(q_n B(t_n, \boldsymbol{\Theta}_n)\right) + \nu\Delta\boldsymbol{v}_n - \nabla q_n \\ &\quad - \nu\Delta\left(\boldsymbol{v}_n - \boldsymbol{u}_{lim}\right) + \nabla(q_n - p_{lim}), \qquad \text{in } \Omega\backslash\overline{\omega^*} \end{aligned}$$

and

$$\nabla\boldsymbol{v}_n : B(t_n, \boldsymbol{\Theta}_n) - \operatorname{div}\boldsymbol{u}_{lim} = \nabla\boldsymbol{v}_n : B(t_n, \boldsymbol{\Theta}_n) - \operatorname{div}\boldsymbol{v}_n + \operatorname{div}\left(\boldsymbol{v}_n - \boldsymbol{u}_{lim}\right) \qquad \text{in } \Omega\backslash\overline{\omega^*}.$$

Let us focus on the second equality. We proceed in the same way for the first one. Since $\boldsymbol{v}_n \to \boldsymbol{u}_{lim}$ in $\mathcal{C}^2(\overline{\Omega\backslash\overline{\omega^*}})$, we obtain, passing to the limit in the previous equality and using Problem (4.3) solved by the couple $(\boldsymbol{v}_n, q_n)$, that

$$\operatorname{div}\boldsymbol{u}_{lim} = 0 \qquad \text{in } \Omega\backslash\overline{\omega^*}.$$

Thus, using the boundary conditions, we obtain that the couple $(\boldsymbol{u}_{lim}, p_{lim})$ solves

$$\begin{cases} -\nu\Delta\boldsymbol{u}_{lim} + \nabla p_{lim} &= \boldsymbol{0} \quad &\text{in } \Omega\backslash\overline{\omega^*} \\ \operatorname{div}\boldsymbol{u}_{lim} &= 0 \quad &\text{in } \Omega\backslash\overline{\omega^*} \\ \boldsymbol{u}_{lim} &= \boldsymbol{g} \quad &\text{on } \partial\Omega \\ \boldsymbol{u}_{lim} &= \boldsymbol{0} \quad &\text{on } \partial\omega^*. \end{cases}$$

The uniqueness of such a solution implies $(\boldsymbol{u}_{lim}, p_{lim}) = (\boldsymbol{u}, p)$ which contradicts the second inequality of (4.5). $\qquad\square$

4.3 Proof of Theorem 4.2

Again, we fix $\boldsymbol{\Theta} \in \mathfrak{U}$ a diffeomorphism with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta_2$ such that $|\boldsymbol{\Theta}(\omega^*)| = |\omega^*|$. As previously, we consider the vector field $\boldsymbol{Y}_{\boldsymbol{\Theta}}$, the associated flow Φ_t and the domains $(\omega_t := \Phi_t(\omega^*))_{t \in [0,1]}$ and we set $m_{\boldsymbol{Y}_{\boldsymbol{\Theta}}} := \boldsymbol{Y}_{\boldsymbol{\Theta}} \cdot \widetilde{\mathbf{n}}$. The constant η_2 is given by Lemma 4.4. We denote by C any constant depending only on $\|\boldsymbol{\Theta}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}$.

We have proved (see (3.6)) that

$$
\mathrm{D}^2 J(\omega_t) \cdot \boldsymbol{Y}_{\boldsymbol{\Theta}} \cdot \boldsymbol{Y}_{\boldsymbol{\Theta}} = \underbrace{-\frac{1}{2} \int_{\partial \omega_t} \nu \left(\nabla(|\mathcal{D}(\boldsymbol{u}_t)|^2) \cdot \boldsymbol{Y}_{\boldsymbol{\Theta}} \right) (\boldsymbol{Y}_{\boldsymbol{\Theta}} \cdot \mathbf{n}_t)}_{=: I_1(t)} + \underbrace{2 \int_{\partial \omega_t} \boldsymbol{u}_t' \cdot (\sigma(\boldsymbol{u}_t', p_t') \mathbf{n}_t)}_{=: I_2(t)} .
$$

This decomposition is similar as the one in J_1 and J_2 for the Hessian at the critical shape. It is natural since the two depends on the deformation parameter $\boldsymbol{Y}_{\boldsymbol{\Theta}}$ through two distinct norms. Hence, we will prove the following two lemmas:

Lemma 4.5. *We have*

$$
|I_1(t) - I_1(0)| \leq C\, w(\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)})\, \|m_{\boldsymbol{Y}_{\boldsymbol{\Theta}}}\|_{\mathrm{L}^2(\partial \omega^*)}^2 .
$$

Lemma 4.6. *We have*

$$
|I_2(t) - I_2(0)| \leq C\, w(\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)})\, \|m_{\boldsymbol{Y}_{\boldsymbol{\Theta}}}\|_{\mathrm{H}^{1/2}(\partial \omega^*)}^2 .
$$

In both lemmas, w denotes the modulus of continuity introduced in Proposition 4.3. Using the previous characterization of $\mathrm{D}^2 J(\omega_t) \cdot \boldsymbol{Y}_{\boldsymbol{\Theta}} \cdot \boldsymbol{Y}_{\boldsymbol{\Theta}}$ and these two lemmas, Theorem 4.2 is proved. Hence, it now suffices to prove these lemmas. To simplify the notations, we set

$$
\eta := \|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} ,
$$

and, in the sequel, we denote all the constants depending only on the data and on the norm $\|\boldsymbol{\Theta}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)}$ by C.

4.3.1 Proof of Lemma 4.5

We know that

$$
\nabla \boldsymbol{u}_t \circ \Phi_t = {}^t[\mathrm{D}\Phi_t]^{-1} \nabla \boldsymbol{v}_t \quad \text{and} \quad \mathcal{D}(\boldsymbol{u}_t) \circ \Phi_t = {}^t[\mathrm{D}\Phi_t]^{-1} \mathcal{D}(\boldsymbol{v}_t).
$$

Hence, replacing $\boldsymbol{u}_t$ by $|\mathcal{D}(\boldsymbol{u}_t)|^2$,

$$
\nabla \left(|\mathcal{D}(\boldsymbol{u}_t)|^2 \right) \circ \Phi_t = {}^t[\mathrm{D}\Phi_t]^{-1} \nabla(|\mathcal{D}(\boldsymbol{u}_t)|^2 \circ \Phi_t) = {}^t[\mathrm{D}\Phi_t]^{-1} \nabla \left(\left| {}^t[\mathrm{D}\Phi_t]^{-1} \mathcal{D}(\boldsymbol{v}_t) \right|^2 \right) .
$$

Thus, using the change of variables $x = \Phi_t y$,

$$
\begin{aligned}
I_1(t) &= \int_{\partial \omega^*} \nu\, {}^t\mathrm{D}\Phi_t^{-1} \nabla \left(\left| {}^t\mathrm{D}\Phi_t^{-1} \mathcal{D}(\boldsymbol{v}_t) \right|^2 \right) \cdot (\boldsymbol{Y}_{\boldsymbol{\Theta}} \circ \Phi_t) \left[(\boldsymbol{Y}_{\boldsymbol{\Theta}} \circ \Phi_t) \cdot (\mathbf{n}_t \circ \Phi_t) \right] J(t) \\
&= \int_{\partial \omega^*} \nu\, |m_{\boldsymbol{Y}_{\boldsymbol{\Theta}}} \circ \Phi_t|^2\, {}^t\mathrm{D}\Phi_t^{-1} \nabla \left(\left| {}^t\mathrm{D}\Phi_t^{-1} \mathcal{D}(\boldsymbol{v}_t) \right|^2 \right) \cdot \mathbf{n}\, (\mathbf{n} \cdot (\mathbf{n}_t \circ \Phi_t))\, J(t), \quad (4.6)
\end{aligned}
$$

where $J(t) := \det \mathrm{D}\Phi_t / \|({}^t\mathrm{D}\Phi_t^{-1})\mathbf{n}\|$ is the surface jacobian. In order to study $I_1(t) - I_1(0)$, let us define

$$
\begin{aligned}
a_0(t) &:= m_Y \circ \Phi_t, & a_1(t) &:= {}^t\mathrm{D}\Phi_t^{-1} \nabla \left(\left| {}^t\mathrm{D}\Phi_t^{-1} \mathcal{D}(\boldsymbol{v}_t) \right|^2 \right) \cdot \mathbf{n}, \\
a_2(t) &:= \mathbf{n} \cdot (\mathbf{n}_t \circ \Phi_t), & a_3(t) &:= J(t).
\end{aligned}
$$

Using the estimate concerning $\mathbf{n}_t \circ \Phi_t$ given in Lemma 2.1, we have

$$
\begin{array}{rclcl}
\|a_2(t)\|_{\mathrm{L}^\infty(\partial\omega^*)} & = & \|\mathbf{n} \cdot (\mathbf{n}_t \circ \Phi_t)\|_{\mathrm{L}^\infty(\partial\omega^*)} & \leq & C \\
\|a_2(t) - a_2(0)\|_{\mathrm{L}^\infty(\partial\omega^*)} & = & \|\mathbf{n} \cdot (\mathbf{n}_t \circ \Phi_t - \mathbf{n})\|_{\mathrm{L}^\infty(\partial\omega^*)} & \leq & C\,\eta.
\end{array}
\tag{4.7}
$$

Moreover, using the estimate concerning $J(t)$ given in Lemma 2.1, we have

$$
\begin{array}{rclcl}
\|a_3(t)\|_{\mathrm{L}^\infty(\partial\omega^*)} & = & \|J(t)\|_{\mathrm{L}^\infty(\partial\omega^*)} & \leq & C \\
\|a_3(t) - a_3(0)\|_{\mathrm{L}^\infty(\partial\omega^*)} & = & \|J(t) - 1\|_{\mathrm{L}^\infty(\partial\omega^*)} & \leq & C\,\eta.
\end{array}
\tag{4.8}
$$

Let us now prove that, for all $t \in [0,1]$,

$$
\|a_1(t)\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C \quad \text{and} \quad \|a_1(t) - a_1(0)\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C\,w(\eta),
\tag{4.9}
$$

where w is the modulus of continuity given in Proposition 4.3. Using the fact that $\boldsymbol{v}_t$ is bounded in $\boldsymbol{\mathcal{C}}^{2,\alpha}(\Omega \backslash \overline{\omega^*})$ (see Lemma 4.4), we have immediately

$$
\|a_1(t)\|_{\mathrm{L}^\infty(\partial\omega^*)} = \left\| {}^t\mathrm{D}\Phi_t^{-1}\nabla\left(\left|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\boldsymbol{v}_t)\right|^2\right) \cdot \mathbf{n} \right\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C.
$$

Let us define $\xi(t) := \nabla\left(\left|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\boldsymbol{v}_t)\right|^2\right)$. Inserting $\xi(t)$, we obtain

$$
\begin{aligned}
\|a_1(t) - a_1(0)\|_{\mathrm{L}^\infty(\partial\omega^*)} &= \left\| \left({}^t\mathrm{D}\Phi_t^{-1} - \mathrm{I}\right)\xi(t) \cdot \mathbf{n} + \left(\xi(t) - \nabla\left(|\mathcal{D}(\boldsymbol{u})|^2\right)\right) \cdot \mathbf{n} \right\|_{\mathrm{L}^\infty(\partial\omega^*)} \\
&\leq C\,\|\xi(t)\|_{\mathrm{L}^\infty(\partial\omega^*)}\left\|{}^t\mathrm{D}\Phi_t^{-1}-\mathrm{I}\right\|_{\mathrm{L}^\infty(\partial\omega^*)} + C\left\|\xi(t)-\nabla\left(|\mathcal{D}(\boldsymbol{u})|^2\right)\right\|_{\mathrm{L}^\infty(\partial\omega^*)} \\
&\leq C\left\|{}^t\mathrm{D}\Phi_t^{-1}-\mathrm{I}\right\|_{\mathrm{L}^\infty(\partial\omega^*)} + C\left\|\xi(t)-\nabla\left(|\mathcal{D}(\boldsymbol{u})|^2\right)\right\|_{\mathrm{L}^\infty(\partial\omega^*)},
\end{aligned}
$$

since $\|\xi(t)\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C$ using (2.3). By Estimate (2.3), we have $\left\|{}^t\mathrm{D}\Phi_t^{-1} - \mathrm{I}\right\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C\eta$ and using the estimate on the norm of $\boldsymbol{v}_t$ given by Proposition 4.3,

$$
\left\|\xi(t) - \nabla\left(|\mathcal{D}(\boldsymbol{u})|^2\right)\right\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq w(\eta).
$$

Hence, we obtain the second estimate of (4.9). Finally, Lemma 2.1 ensures that

$$
\begin{array}{l}
\|a_0(t)\|_{\mathrm{L}^2(\partial\omega^*)} = \|m_{\boldsymbol{Y}_\Theta} \circ \Phi_t\|_{\mathrm{L}^2(\partial\omega^*)} \leq C\,\|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{L}^2(\partial\omega^*)} \\
\|a_0(t) - a_0(0)\|_{\mathrm{L}^2(\partial\omega^*)} \leq C\,\eta\,\|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{L}^2(\partial\omega^*)}.
\end{array}
\tag{4.10}
$$

According to the expression (4.6) of I_1 and the definition of a_i ($i = 0, \dots, 3$),

$$
I_1(t) - I_1(0) = \int_{\partial\omega^*} \nu\left(a_0^2(t) - a_0^2(0)\right)\prod_{i=1}^3 a_i(t) + \int_{\partial\omega^*} \nu\,a_0^2(t)\left(\prod_{i=1}^3 a_i(t) - \prod_{i=1}^3 a_i(0)\right).
\tag{4.11}
$$

The last integral of (4.11) is bounded by

$$
C\,\|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{L}^2(\partial\omega^*)}^2 \left\|\prod_{i=1}^3 a_i(t) - \prod_{i=1}^3 a_i(0)\right\|_{\mathrm{L}^\infty(\partial\omega^*)}.
$$

Using the estimates on the norms of a_i ($i = 1, \dots, 3$) given by (4.7), (4.8) and (4.9) and the fact that $w(\eta) \geq C\,\eta$ by Assumption (4.4), we have

$$
\left\|\prod_{i=1}^3 a_i(t) - \prod_{i=1}^3 a_i(0)\right\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C\sum_{i=1}^3 \|a_i(t) - a_i(0)\|_{\mathrm{L}^\infty(\partial\omega^*)} \leq C(w(\eta) + \eta) \leq C\,w(\eta).
$$

15

Moreover, the first integral of (4.11) is bounded by

$$\left\| \prod_{i=1}^{3} a_i(t) \right\|_{L^\infty(\partial\omega^*)} \|a_0(t) - a_0(0)\|_{L^2(\partial\omega^*)} \|a_0(t) + a_0(0)\|_{L^2(\partial\omega^*)}$$

and using the estimates on the norms of a_i ($i = 0, \ldots, 3$) given by (4.7), (4.8), (4.9) and (4.10) and the remark that $w(\eta) \geq C\,\eta$, it is bounded by

$$C\,w(\eta)\,\|m_{\boldsymbol{Y}_\Theta}\|^2_{L^2(\partial\omega^*)},$$

which concludes the proof.

4.3.2   Proof of Lemma 4.6

Since $\boldsymbol{u}'_t = -\partial_{\mathbf{n}_t}\boldsymbol{u}_t(\boldsymbol{Y}_\Theta \cdot \mathbf{n}_t)$ on $\partial\omega_t$,

$$\boldsymbol{u}'_t = -m_{\boldsymbol{Y}_\Theta}\partial_{\mathbf{n}_t}\boldsymbol{u}_t(\mathbf{n}_t \cdot \widetilde{\mathbf{n}}) \qquad \text{on } \partial\omega_t. \tag{4.12}$$

Let us define the following Steklov-Poincaré operator corresponding to D_N (introduced in Lemma 3.1) on $\partial\omega_t$:

$$C_t : \quad \mathbf{H}^{1/2}(\partial\omega_t) \quad \longrightarrow \quad \mathbf{H}^{-1/2}(\partial\omega_t)$$
$$\boldsymbol{z}_t \quad \longmapsto \quad C_t(\boldsymbol{z}_t) := \sigma(\boldsymbol{Z}_t, \Pi_t)\mathbf{n}_t,$$

where $(\boldsymbol{Z}_t, \Pi_t) \in \mathbf{H}^1(\Omega\backslash\overline{\omega_t}) \times L^2(\Omega\backslash\overline{\omega_t})$ solves

$$\begin{cases} -\mathrm{div}\,(\sigma(\boldsymbol{Z}_t, \Pi_t)) &=& \mathbf{0} & \text{in } \Omega\backslash\overline{\omega_t} \\ \mathrm{div}\,\boldsymbol{Z}_t &=& 0 & \text{in } \Omega\backslash\overline{\omega_t} \\ \boldsymbol{Z}_t &=& \mathbf{0} & \text{on } \partial\Omega \\ \boldsymbol{Z}_t &=& \boldsymbol{z}_t & \text{on } \partial\omega_t. \end{cases} \tag{4.13}$$

We obviously have

$$\frac{1}{2}\nu\,\|\mathcal{D}(\boldsymbol{Z}_t)\|^2_{L^2(\Omega\backslash\overline{\omega_t})} = \langle C_t(\boldsymbol{z}_t)\,,\,\boldsymbol{z}_t\rangle_{\partial\omega_t}. \tag{4.14}$$

Let us define $\boldsymbol{z}_t := -m_{\boldsymbol{Y}_\Theta}\partial_{\mathbf{n}_t}\boldsymbol{u}_t(\mathbf{n}_t \cdot \widetilde{\mathbf{n}})$. Thus, according to the boundary condition of $\boldsymbol{u}'_t$ (see (4.12)), $\sigma(\boldsymbol{u}'_t, p'_t)\mathbf{n}_t = C_t(\boldsymbol{z}_t)$ and, using the definition of I_2, we obtain

$$I_2(t) = 2\int_{\partial\omega_t} \partial_{\mathbf{n}_t}\boldsymbol{u}_t\,C_t(z_t)(\boldsymbol{Y}_\Theta \cdot \mathbf{n}_t).$$

Hence, using Estimate (4.14) on the norm of $\mathcal{D}(\cdot)$, we get

$$I_2(t) = -2\int_{\partial\omega_t} \boldsymbol{z}_t C_t(\boldsymbol{z}_t) = -\nu\,\|\mathcal{D}(\boldsymbol{Z}_t)\|^2_{\mathbf{L}^2(\Omega\backslash\overline{\omega_t})},$$

where $(\boldsymbol{Z}_t, \Pi_t) \in \mathbf{H}^1(\Omega\backslash\overline{\omega_t}) \times L^2(\Omega\backslash\overline{\omega_t})$ solves (4.13).

We want now to estimate

$$\begin{aligned} I_2(t) - I_2(0) &=& \int_{\Omega\backslash\overline{\omega^*}} \nu\,|\mathcal{D}(\boldsymbol{Z}_0)|^2 - \int_{\Omega\backslash\overline{\omega_t}} \nu\,|\mathcal{D}(\boldsymbol{Z}_t)|^2 \\ &=& \int_{\Omega\backslash\overline{\omega^*}} \nu\,|\mathcal{D}(\boldsymbol{Z}_0)|^2 - \int_{\Omega\backslash\overline{\omega^*}} \nu\,\left|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\boldsymbol{Z}}_t)\right|^2 Jac_t, \end{aligned}$$

where $\widetilde{\boldsymbol{Z}}_t := \boldsymbol{Z}_t \circ \Phi_t$. Since $\mathrm{div}\,\boldsymbol{Y}_\Theta = 0$, we get $Jac_t = 1$. We assume for a while that the following lemma is proved:

Lemma 4.7. *For all $t \in [0,1]$,*

$$
\begin{aligned}
\|\widetilde{z}_t\|_{\mathbf{H}^{1/2}(\partial\omega^*)} &\leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)}, & \|\widetilde{z}_t - z_0\|_{\mathbf{H}^{1/2}(\partial\omega^*)} &\leq Cw(\eta) \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)}, \\
\left\|\widetilde{\mathbf{Z}}_t\right\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} &\leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} \quad and \quad & \left\|\widetilde{\mathbf{Z}}_t - \mathbf{Z}_0\right\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} &\leq Cw(\eta) \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)}.
\end{aligned}
$$

Using this lemma and the fact that $\left\|{}^t\mathrm{D}\Phi_t^{-1}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C$ and $\left\|{}^t\mathrm{D}\Phi_t^{-1} - \mathrm{I}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C\eta$ (see Estimate (2.3)), we have

$$
\begin{aligned}
|I_2(t) - I_2(0)| &= \int_{\Omega\backslash\overline{\omega^*}} \nu \left|\mathcal{D}(\mathbf{Z}_0)\right|^2 - \int_{\Omega\backslash\overline{\omega^*}} \nu \left|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\mathbf{Z}}_t)\right|^2 \\
&\leq C \left\|\mathcal{D}(\mathbf{Z}_0) + {}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\mathbf{Z}}_t)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \left\|\mathcal{D}(\mathbf{Z}_0) - {}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\mathbf{Z}}_t)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \\
&\leq C \left(\|\mathcal{D}(\mathbf{Z}_0)\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} + \left\|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\mathbf{Z}}_t)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \right) \\
&\qquad \left(\left\|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\mathbf{Z}}_t) - {}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\mathbf{Z}_0)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} + \left\|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\mathbf{Z}_0) - \mathcal{D}(\mathbf{Z}_0)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \right) \\
&\leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} \left(\left\|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\widetilde{\mathbf{Z}}_t) - {}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\mathbf{Z}_0)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \right. \\
&\qquad \left. + \left\|{}^t\mathrm{D}\Phi_t^{-1}\mathcal{D}(\mathbf{Z}_0) - \mathcal{D}(\mathbf{Z}_0)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \right) \\
&\leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} \left(\left\|\mathcal{D}(\widetilde{\mathbf{Z}}_t) - \mathcal{D}(\mathbf{Z}_0)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} + C\eta \|\mathcal{D}(\mathbf{Z}_0)\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \right) \\
&\leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} \left(w(\eta) \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} + \eta \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} \right).
\end{aligned}
$$

The proof is completed. It remains to prove Lemma 4.7.

Proof of Lemma 4.7. We recall that

$$
\mathbf{z}_t := -m_{\mathbf{Y}_\Theta} \partial_{\mathbf{n}_t} \mathbf{u}_t (\mathbf{n}_t \cdot \widetilde{\mathbf{n}}) \quad \text{and} \quad \widetilde{\mathbf{z}}_t := \mathbf{z}_t \circ \Phi_t.
$$

We use the following product lemma:

$$
\forall \mathbf{v} \in \mathbf{H}^{1/2}(\partial\omega^*), \ \mathbf{w} \in \boldsymbol{\mathcal{C}}^1(\partial\omega^*), \ \|\mathbf{v}\,\mathbf{w}\|_{\mathbf{H}^{1/2}(\partial\omega^*)} \leq C \|\mathbf{v}\|_{\mathbf{H}^{1/2}(\partial\omega^*)} \|\mathbf{w}\|_{\boldsymbol{\mathcal{C}}^1(\partial\omega^*)}
$$

for some constant C depending only on $\Omega\backslash\overline{\omega}$ to get

$$
\|\widetilde{\mathbf{z}}_t\|_{\mathbf{H}^{1/2}(\partial\omega^*)} \leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)} \|\partial_{\mathbf{n}_t} \mathbf{u}_t (\mathbf{n}_t \cdot \mathbf{n})\|_{\boldsymbol{\mathcal{C}}^1(\partial\omega^*)}.
$$

Then, using the estimate on the norm of $\mathbf{n}_t$ given by Lemma 2.1 and the estimate on the norm of $\mathbf{v}_t$ given by Proposition 4.3, we obtain the first estimate of the lemma:

$$
\|\widetilde{\mathbf{z}}_t\|_{\mathbf{H}^{1/2}(\partial\omega^*)} \leq C \|m_{\mathbf{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial\omega^*)}. \tag{4.15}
$$

Defining $\widetilde{\mathbf{n}}_t := \mathbf{n}_t \circ \Phi_t$, we have

$$
\widetilde{\mathbf{z}}_t - \mathbf{z}_0 = \widetilde{\mathbf{z}}_t - m_{\mathbf{Y}_\Theta} \partial_{\mathbf{n}} \mathbf{u} = m_{\mathbf{Y}_\Theta} \left(\partial_{\mathbf{n}} \mathbf{u} - \partial_{\mathbf{n}_t} \mathbf{u}_t \circ \Phi_t (\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \right) \qquad \text{on } \partial\omega^*.
$$

However,

$$
\begin{aligned}
\|\partial_{\mathbf{n}} \mathbf{u} - \partial_{\mathbf{n}_t} \mathbf{u}_t \circ \Phi_t (\widetilde{\mathbf{n}}_t \cdot \mathbf{n})\|_{\boldsymbol{\mathcal{C}}^1(\partial\omega^*)} &\leq \|\partial_{\mathbf{n}} \mathbf{u} \left(1 - (\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \right)\|_{\boldsymbol{\mathcal{C}}^1(\partial\omega^*)} \\
&\qquad + \|(\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \left(\partial_{\mathbf{n}} \mathbf{u} - \partial_{\mathbf{n}_t} \mathbf{u}_t \circ \Phi_t \right)\|_{\boldsymbol{\mathcal{C}}^1(\partial\omega^*)}.
\end{aligned}
$$

Moreover, using the estimate on the norm of $\mathbf{n}_t$ given by Lemma 2.1, we have

$$\left\| \partial_{\mathbf{n}} \boldsymbol{u} \left(1 - (\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \right) \right\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)} \le C \eta,$$

since $\|\partial_{\mathbf{n}} \boldsymbol{u}\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)} \le C$. Now, let us bound $\left\| (\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \left(\partial_{\mathbf{n}} \boldsymbol{u} - \partial_{\mathbf{n}_t} \boldsymbol{u}_t \circ \Phi_t \right) \right\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)}$. Using again Lemma 2.1, Estimate (2.3) and Proposition 4.3, we obtain

$$\left\| (\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \left(\partial_{\mathbf{n}} \boldsymbol{u} - \partial_{\mathbf{n}_t} \boldsymbol{u}_t \circ \Phi_t \right) \right\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)} = \left\| (\widetilde{\mathbf{n}}_t \cdot \mathbf{n}) \left(\partial_{\mathbf{n}} \boldsymbol{u} - {}^t\mathrm{D}\Phi_t^{-1} \nabla \boldsymbol{v}_t \widetilde{\mathbf{n}}_t \right) \right\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)}$$

$$\le C \left(\|\nabla \boldsymbol{u} (\mathbf{n}_t - \mathbf{n})\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)} + \left\| \left(\nabla \boldsymbol{u} - {}^t\mathrm{D}\Phi_t^{-1} \nabla \boldsymbol{v}_t \right) \mathbf{n}_t \right\|_{\boldsymbol{\mathcal{C}}^1(\partial \omega^*)} \right) \le C w(\eta),$$

since we can assume $\eta \le C w(\eta)$ (see (4.4)). Hence, the second inequality of the lemma is proved:

$$\|\widetilde{\boldsymbol{z}}_t - \boldsymbol{z}_0\|_{\mathbf{H}^{1/2}(\partial \omega^*)} \le C w(\eta) \|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial \omega^*)}. \tag{4.16}$$

Since $\boldsymbol{Z}_0$ solves the Stokes system with $\boldsymbol{z}_0$ as interior boundary condition, the following energy estimate holds using the classical energy estimate concerning the Stokes system (see for example the books [14, 12]) and the first estimate (4.15)

$$\|\boldsymbol{Z}_0\|_{\mathbf{H}^1(\Omega \backslash \overline{\omega^*})} + \|\Pi_0\|_{\mathrm{L}^2(\Omega \backslash \overline{\omega^*})} \le C \|\boldsymbol{z}_0\|_{\mathbf{H}^{1/2}(\partial \omega^*)} \le C \|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial \omega^*)}. \tag{4.17}$$

With this inequality, it remains to prove the inequality concerning $\left\| \widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0 \right\|_{\mathbf{H}^1(\Omega \backslash \overline{\omega^*})}$. Indeed, we will then have

$$\left\| \widetilde{\boldsymbol{Z}}_t \right\|_{\mathbf{H}^1(\Omega \backslash \overline{\omega^*})} \le C w(\eta) \|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial \omega^*)} + \|\boldsymbol{Z}_0\|_{\mathbf{H}^1(\Omega \backslash \overline{\omega^*})} \le C \|m_{\boldsymbol{Y}_\Theta}\|_{\mathrm{H}^{1/2}(\partial \omega^*)}.$$

The couple $(\boldsymbol{Z}_t, \Pi_t) \in \mathbf{H}^1(\Omega \backslash \overline{\omega_t}) \times \mathrm{L}^2(\Omega \backslash \overline{\omega_t})$ solves (4.13). Thus, defining $\widetilde{\boldsymbol{Z}}_t := \boldsymbol{Z}_t \circ \Phi_t$ and $\widetilde{\Pi}_t := \Pi \circ \Phi_t$, we prove that

$$\begin{cases} -\mathrm{div}\left(\nabla \widetilde{\boldsymbol{Z}}_t A(t) \right) + \mathrm{div}\left(\widetilde{\Pi}_t B(t) \right) &= \boldsymbol{0} & \text{in } \Omega \backslash \overline{\omega^*} \\ \nabla \widetilde{\boldsymbol{Z}}_t : B(t) &= 0 & \text{in } \Omega \backslash \overline{\omega^*} \\ \widetilde{\boldsymbol{Z}}_t &= \boldsymbol{0} & \text{on } \partial \Omega \\ \widetilde{\boldsymbol{Z}}_t &= \widetilde{\boldsymbol{z}}_t & \text{on } \partial \omega^*. \end{cases}$$

Using the problem solved by $(\boldsymbol{Z}_0, \Pi_0)$, this problem can be rewritten as

$$\begin{cases} \nu \Delta(\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0) - \nabla(\widetilde{\Pi}_t - \Pi_0) &= L_1(t)\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t \right) - S\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t \right) & \text{in } \Omega \backslash \overline{\omega^*} \\ -\mathrm{div}\left(\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0 \right) &= L_2(t)(\widetilde{\boldsymbol{Z}}_t) - \mathrm{div}\left(\widetilde{\boldsymbol{Z}}_t \right) & \text{in } \Omega \backslash \overline{\omega^*} \\ \widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0 &= \boldsymbol{0} & \text{on } \partial \Omega \\ \widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0 &= \widetilde{\boldsymbol{z}}_t - \boldsymbol{z}_0 & \text{on } \partial \omega^*, \end{cases}$$

with

$$\begin{aligned} L_1(t)\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t \right) &:= -\mathrm{div}\left(\nabla \widetilde{\boldsymbol{Z}}_t A(t) \right) + \mathrm{div}\left(\widetilde{\Pi}_t B(t) \right) \\ L_2(t)(\widetilde{\boldsymbol{Z}}_t) &:= \nabla \widetilde{\boldsymbol{Z}}_t : B(t) \\ S\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t \right) &:= -\nu \Delta(\widetilde{\boldsymbol{Z}}_t) + \nabla(\widetilde{\Pi}_t). \end{aligned}$$

Thus, using the energy estimate,

$$\left\|\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0\right\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} + \left\|\widetilde{\Pi}_t - \Pi_0\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \leq C\bigg(\|\widetilde{\boldsymbol{z}}_t - \boldsymbol{z}_0\|_{\mathbf{H}^{1/2}(\partial\omega^*)}$$
$$\left\|(L_1(t) - S)\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t\right)\right\|_{\mathbf{H}^{-1}(\Omega\backslash\overline{\omega^*})} + \left\|(L_2(t) - \mathrm{div}\,)\left(\widetilde{\boldsymbol{Z}}_t\right)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \bigg). \quad (4.18)$$

We have

$$(L_1(t) - S)\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t\right) = (L_1(t) - S)\left(\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0, \widetilde{\Pi}_t - \Pi_0\right) + (L_1(t) - S)\left(\widetilde{\boldsymbol{Z}}_0, \widetilde{\Pi}_0\right)$$
$$(L_2(t) - \mathrm{div}\,)\left(\widetilde{\boldsymbol{Z}}_t\right) = (L_2(t) - \mathrm{div}\,)\left(\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0\right) + (L_2(t) - \mathrm{div}\,)\left(\widetilde{\boldsymbol{Z}}_0\right).$$

Let us assume for a while

$$\begin{aligned}
\|(L_1(t) - S)\,(\boldsymbol{w}, q)\|_{\mathbf{H}^{-1}(\Omega\backslash\overline{\omega^*})} &\leq C\eta\left(\|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} + \|q\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})}\right) \\
\|(L_2(t) - \mathrm{div}\,)\,(\boldsymbol{w})\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} &\leq C\eta\,\|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})}.
\end{aligned} \quad (4.19)$$

Then,

$$\left\|(L_1(t) - S)\left(\widetilde{\boldsymbol{Z}}_t, \widetilde{\Pi}_t\right)\right\|_{\mathbf{H}^{-1}(\Omega\backslash\overline{\omega^*})} + \left\|(L_2(t) - \mathrm{div}\,)\left(\widetilde{\boldsymbol{Z}}_t\right)\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})}$$
$$\leq C\eta\left(\left\|\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0\right\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} + \left\|\widetilde{\Pi}_t - \Pi_0\right\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})}\right) + C\eta\,\|\boldsymbol{Z}_0\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})}.$$

Then, using this inequality in Estimate (4.18) and Estimates (4.16) and (4.17) on the norms of $\widetilde{\boldsymbol{z}}_t - \boldsymbol{z}_0$ and $\boldsymbol{Z}_0$, we obtain

$$\left\|\widetilde{\boldsymbol{Z}}_t - \boldsymbol{Z}_0\right\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} (1 - C\eta) \leq Cw(\eta)\,\|\boldsymbol{V}\|_{\mathbf{H}^{1/2}(\partial\omega^*)},$$

which concludes the proof.

It remains to prove Estimate (4.19). Let us consider $\boldsymbol{\Phi} \in \mathbf{H}_0^1(\Omega\backslash\overline{\omega^*})$. We have

$$\left|\int_{\Omega\backslash\overline{\omega^*}} (L_1(t) - S)\,(\boldsymbol{w}, q) \cdot \boldsymbol{\Phi}\right| = \left|\int_{\Omega\backslash\overline{\omega^*}} (-\mathrm{div}\,(\nabla\boldsymbol{w}A(t)) + \mathrm{div}\,(qB(t)) + \nu\Delta\boldsymbol{w} - \nabla q) \cdot \boldsymbol{\Phi}\right|$$

$$= \left|\int_{\Omega\backslash\overline{\omega^*}} ((\nabla\boldsymbol{w}A(t)) + qB(t) + \nu\nabla\boldsymbol{w} - q\mathrm{I}) : \nabla\boldsymbol{\Phi}\right|$$

$$\leq C\,\|\boldsymbol{\Phi}\|_{\mathbf{H}_0^1(\Omega\backslash\overline{\omega^*})} \Big(\|A(t)\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})}$$

$$+ \|q\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \|B(t)\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} + \|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} + \|q\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})} \Big)$$

$$\leq C\,\|\boldsymbol{\Phi}\|_{\mathbf{H}_0^1(\Omega\backslash\overline{\omega^*})}\,\eta\left(\|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})} + \|q\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})}\right)$$

using Lemma 2.1 and Estimate (2.3). We proceed in the same way to prove that, for all $\xi \in \mathrm{L}^2(\Omega\backslash\overline{\omega^*})$,

$$\left|\int_{\Omega\backslash\overline{\omega^*}} (L_2(t) - \mathrm{div}\,)\,(\boldsymbol{w})\,\xi\right| \leq C\,\|\xi\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega^*})}\,\eta\,\|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega^*})},$$

which concludes the proof of Estimate (4.19). $\qquad\square$

5 Precised stability estimate

The main result of this section is a purely geometrical result valid for small smooth perturbations of a domain.

Proposition 5.1. *In the assumptions of Section 2, there is a constant C that depends only on Ω such that*

$$\mathcal{L}^d(\Omega \Delta \Omega_{per}) \ \leq \ C\|h\|_{\mathrm{L}^1}. \tag{5.1}$$

Proof. We first consider the case of a smooth positive deformation function h and prove (5.1) in this first case. Since h is positive, Ω is a subset of Ω_{per} and hence

$$\mathcal{L}^d(\Omega \Delta \Omega_{per}) = \mathcal{L}^d(\Omega_{per}) - \mathcal{L}^d(\Omega).$$

To this function, is associated the vector field $\boldsymbol{V}$ by the construction (2.1) of subsection 2.2 (in general or by (2.2) if a volume constraint has to satisfied). We then apply the classical Lemma about variations of the volume

$$\frac{d}{dt}\, \mathcal{L}^d(\Omega_t) \ = \ \int_{\partial \Omega_t} \boldsymbol{V} \cdot \mathbf{n}_t \ d\sigma_{\partial \Omega_t}(y)$$

to get along the flow of the field $\boldsymbol{V}$:

$$
\begin{aligned}
\mathcal{L}^d(\Omega_{per}) - \mathcal{L}^d(\Omega) \ &= \ \int_0^1 \int_{\partial \Omega_t} \boldsymbol{V}(y) \cdot \mathbf{n}_t(y) \ d\sigma_{\partial \Omega_t}(y) \ dt \\
&= \ \int_0^1 \int_{\partial \Omega_t} h(p_{\partial \Omega}(y)) \, \mathbf{n}(p_{\partial \Omega}(y)) \cdot \mathbf{n}_t(y) \ d\sigma_{\partial \Omega_t}(y) \ dt \\
&= \ \int_0^1 \int_{\partial \Omega} h(x)\mathbf{n}(x) \cdot \mathbf{n}_t(x + t\boldsymbol{V}(x))J_t(x) \ d\sigma_{\partial \Omega}(x) \ dt \\
&= \ \int_{\partial \Omega} h(x) \int_0^1 \mathbf{n}(x) \cdot \mathbf{n}_t(x + t\boldsymbol{V}(x))J_t(x) \ dt \ d\sigma_{\partial \Omega}(x) \\
&= \ \int_{\partial \Omega} h(x) \int_0^1 \mathbf{n}(x + t\boldsymbol{V}(x)) \cdot \mathbf{n}_t(x + t\boldsymbol{V}(x))J_t(x) \ dt \ d\sigma_{\partial \Omega}(x),
\end{aligned}
$$

where J_t is the surface jacobian. We estimate the middle term

$$\int_0^1 \mathbf{n}(x + t\boldsymbol{V}(x)) \cdot \mathbf{n}_t(x + t\boldsymbol{V}(x))J_t(x) \ dt$$

thanks to Lemma 2.1 and conclude. The same proof works for smooth negative deformation field, then by density to any signed Lipschitz deformation.

For a general deformation, expressed with a smooth normal deformation h, we split it into $h = h^+ - h^-$ its positive and negative part and define $\boldsymbol{V}^+$ and $\boldsymbol{V}^-$ by the construction (2.1). Then it suffices to notice that $\Omega_{per} = \Phi_1(\boldsymbol{V}^-)[\Phi_1(\boldsymbol{V}^+)[\Omega]]$: the idea is to first apply the field $\boldsymbol{V}^+$ to match the inflating areas then the field $\boldsymbol{V}^-$ to contract the domain. $\qquad \square$

Now, in the context of our problem, Cauchy Schwarz inequality provides

$$\|h\|_{\mathrm{L}^1(\partial \omega^*)} \leq |\partial \omega^*|^{1/2}\|h\|_{\mathrm{L}^2(\partial \omega^*)} \leq |\partial \omega^*|^{1/2}\|h\|_{\mathrm{H}^{1/2}(\partial \omega^*)},$$

so that our previous stability estimates (4.2) provides the existence of a non negative real η and of a constant C such that for any diffeomorphism $\boldsymbol{\Theta} \in \mathfrak{U}$ with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta$ and $|\boldsymbol{\Theta}(\Omega)| = |\Omega|$, it holds

$$J(\omega) \geq J(\omega^*) + \frac{C}{4} \|\boldsymbol{Y}_{\Theta} \cdot \mathbf{n}\|_{H^{1/2}(\partial\omega^*)}^2 \geq J(\omega^*) + C \, \mathcal{L}^d(\omega^* \Delta \Theta(\omega^*))^2,$$

where $\boldsymbol{Y}_{\Theta}$ is defined in subsection 2.2. We have shown the precised estimate:

Theorem 5.2. *If ω^* is a critical shape for J where (4.1) holds, there exists $\eta > 0$ and $C > 0$ depending only on Ω and ω^* such that, for all $\boldsymbol{\Theta} \in \mathfrak{U}$ with $\|\boldsymbol{\Theta} - \mathbf{I}\|_{\mathcal{C}^{2,1}(\mathbb{R}^d)} < \eta$, $|\boldsymbol{\Theta}(\omega^*)| = |\omega^*|$ and $\boldsymbol{\Theta} \neq \mathbf{I}$,*

$$J(\boldsymbol{\Theta}(\omega^*)) > J(\omega^*) + C \, \mathcal{L}^d(\omega^* \Delta \Theta(\omega^*))^2.$$

Let us make a final comment: this result can be understood as a quantitative estimate of the deviation from minimality for sets close to ω^* in a strong sense (here the $\mathcal{C}^{2,\alpha}$ norm). Similar results have been introduced recently. The present work corresponds to the first step in extending to the drag functional ideas and result on isoperimetric problems, see [1]. In a second step, using additional regularity properties provided by the perimeter, Acerbi *et al.* manage to extend this result on smooth domains to less regular sets.

6 Conclusion

This paper is, in some sense, a generalization to a vectorial of some previous work (see [10, 9]) where the stability of critical shapes is investigated for the scalar case. Indeed, we here focused on the minimization problem of the energy dissipated by a newtonian and incompressible fluid driven by the Stokes law. Then, we gave a sufficient condition for the shape Hessian to be coercive in a weaker norm than the differentiability norm. However, we showed that this coercivity is sufficient to prove that such a critical shape is a local minimizer. Finally, we obtained a lower bound of the variations of the drag in terms of the symmetric difference of the domain.

This paper may be extended to the case where the fluid motion is assumed to be governed by the (stationary) Navier-Stokes equations. The most difficult part in order to obtain analogous results seems to be the coercivity of the shape Hessian. Moreover, the present work can be adapted to other classical boundary conditions. Finally, it should be possible to adapt these results to unbounded domain. However, due to the approach using derivatives, removing the regularity assumptions on the boundary and the deformations seems out of reach without new ideas.

References

[1] E. Acerbi, N. Fusco, and M. Morini. Minimality via second variation for a nonlocal isoperimetric problem. *preprint*, 2011.

[2] L. Afraites, M. Dambrine, and D. Kateb. On second order shape optimization methods for electrical impedance tomography. *SIAM J. Control Optim.*, 47(3):1556–1590, 2008.

[3] M. Badra, F. Caubet, and M. Dambrine. Detecting an obstacle immersed in a fluid by shape optimization methods. *Math. Models Methods Appl. Sci.*, 21(10):2069–2101, 2011.

[4] J. A. Bello, E. Fernández-Cara, J. Lemoine, and J. Simon. The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes flow. *SIAM J. Control Optim.*, 35(2):626–640, 1997.

[5] J. A. Bello, E. Fernández Cara, and J. Simon. Optimal shape design for Navier-Stokes flow. In *System modelling and optimization (Zurich, 1991)*, volume 180 of *Lecture Notes in Control and Inform. Sci.*, pages 481–489. Springer, Berlin, 1992.

[6] J. A. Bello, E. Fernández Cara, and J. Simon. The variation of the drag with respect to the domain in Navier-Stokes flow. In *Optimization, optimal control and partial differential equations (Iaşi, 1992)*, volume 107 of *Internat. Ser. Numer. Math.*, pages 287–296. Birkhäuser, Basel, 1992.

[7] F. Caubet. Instability of an inverse problem for the stationary Navier-Stokes equations. *HAL-00696174*, 2012. (submitted).

[8] F. Caubet, M. Dambrine, D. Kateb, and C. Z. Timimoun. A Kohn-Vogelius formulation to detect an obstacle immersed in a fluid. *HAL-00678036*, 2012. (submitted).

[9] M. Dambrine. On variations of the shape Hessian and sufficient conditions for the stability of critical shapes. *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 96(1):95–121, 2002.

[10] M. Dambrine and M. Pierre. About stability of equilibrium shapes. *M2AN Math. Model. Numer. Anal.*, 34(4):811–834, 2000.

[11] J. Descloux. On the two dimensional magnetic shaping problem without surface tension. *Report, Analysis and numerical analysis*, (07.90), 1990. Ecole polytechnique fédérale de Lausanne.

[12] C. Fabre and G. Lebeau. Prolongement unique des solutions de l'equation de Stokes. *Comm. Partial Differential Equations*, 21(3-4):573–596, 1996.

[13] N. Fusco and M. Morini. Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of solutions. *Arch. Ration. Mech. Anal.*, 203(1):247–327, 2012.

[14] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations.*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[15] A. Henrot and M. Pierre. *Variation et optimisation de formes*, volume 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].

[16] O. A. Oleĭnik, A. S. Shamaev, and G. A. Yosifian. *Mathematical problems in elasticity and homogenization*, volume 26 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1992.

[17] M. Schäfer, S. Turek, and R. Rannacher. Evaluation of a CFD benchmark for laminar flows. *Proceedings ENUMATH 1997*, (Heidelberg, Germany, 28 September–3 October 1997. World Scientific: Singapore):549–563, 1998.

[18] J. Simon. Differentiation with respect to the domain in boundary value problems. *Numer. Funct. Anal. Optim.*, 2(7-8):649–687 (1981), 1980.

[19] J. Simon. Second variations for domain optimization problems. In *Control and estimation of distributed parameter systems (Vorau, 1988)*, volume 91 of *Internat. Ser. Numer. Math.*, pages 361–378. Birkhäuser, Basel, 1989.

[20] J. Simon. Domain variation for drag in Stokes flow. In *Control theory of distributed parameter systems and applications (Shanghai, 1990)*, volume 159 of *Lecture Notes in Control and Inform. Sci.*, pages 28–42. Springer, Berlin, 1991.

[21] J. Sokołowski and J.-P. Zolésio. *Introduction to shape optimization*, volume 16 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1992. Shape sensitivity analysis.